Optimal Time-Varying Heat Transfer in Multilayered Packages With Arbitrary Heat Generations and Contact Resistance

Integrating the cooling systems of power electronics and electric machines (PEEMs) with other existing vehicle thermal management systems is an innovative technology for the next-generation hybrid electric vehicles (HEVs). As such, the reliability of PEEM must be assured under different dynamic duty cycles. Accumulation of excessive heat within the multilayered packages of PEEMs, due to the thermal contact resistance between the layers and variable temperature of the coolant, is the main challenge that needs to be addressed over a transient thermal duty cycle. Accordingly, a new analytical model is developed to predict transient heat diffusion inside multilayered composite packages. It is assumed that the composite exchanges heat via convection and radiation mechanisms with the surrounding fluid whose temperature varies arbitrarily over time (thermal duty cycle). As such, a time-dependent conjugate convection and radiation heat transfer is considered for the outer-surface. Moreover, arbitrary heat generation inside the layers and thermal contact resistances between the layers are taken into account. New closed-form relationships are developed to calculate the temperature distribution inside multilayered media. The present model is used to find an optimum value for the angular frequency of the surrounding fluid temperature to maximize the interfacial heat flux of the composite media; up to 10% higher interfacial heat dissipation rate compared to constant fluid-temperature case. An independent numerical simulation is also performed using Consou Multiphysics; the maximum relative difference between the obtained numerical data and the analytical model is less than 6%.

Keywords: multilayered composite systems, transient conduction, thermal contact resistance, electronic packages, dynamic thermal response, integrated thermal management

1 Introduction

Integration of component thermal management technologies for new propulsion systems is a key to developing innovative technology for the next-generation HEVs, electric vehicles (EVs), and fuel cell vehicles (FCVs). Current hybrid systems use a separate low-temperature liquid cooling loop for the PEEM [1,2]. However, utilizing an integrated cooling loop for an HEV addresses the cost, weight, size, and fuel consumption [2–4]. As such, the cooling systems of PEEM can be integrated with (i) the air conditioning system via a low-temperature coolant loop and (ii) the internal combustion engine (ICE) via a high-temperature cooling system. Typically, steady-state scenarios were considered for the thermal management of conventional vehicles [3,5]. However, within the context of “integrated thermal management,” the evaluation over a transient thermal duty cycle is important. This is attributed to the fact that certain components may not experience peak thermal loads over steady-state cases or at the same time during a vehicle driving cycle [2]. Misalignment of the peak heat loads leads to variable time-dependent coolant temperature for PEEM over a transient duty cycle. For instance, for the above-mentioned high-temperature cooling loop of PEEM and ICE, the coolant temperature of the electric machine was reported to fluctuate between 80°C and 110°C, depending upon the functionality of ICE during the studied duty cycle [3].

To successfully integrate dynamic PEEM cooling systems concept into vehicle applications, the thermal limitations of the semiconductor devices must be addressed [6]. For critical semiconductor devices such as insulated gate bipolar transistors (IGBTs) in an HEV, McGlen et al. [7] predicted heat fluxes of 150 – 200(W/cm²) and pulsed transient heat loads with heat fluxes up to 400(W/cm²). According to Samsung technologists, the next-generation semiconductor technology costs about $10 billion to create [6]. Alternatively, utilization of multilayered packages is recognized as an innovative technique for the thermal management of semiconductor devices, which also results in improved performance through lowering of signal delays and increasing processing speed. However, due to the dynamic unsteady characteristics of the power electronics inside HEVs/ EVs/FCVs, accumulation of excessive heat within the multilayered packages as a result of thermal contact resistance between the layers as well as variable temperature of the coolant is the main challenge that needs to be addressed [6–12]. One important aspect of the thermal design of such dynamic multilayered systems is the ability to obtain an accurate transient temperature solution of the packages beforehand in order to sustain the reliability of the packages albeit for a more simplified configuration.

Transient heat conduction in multilayered packages has been the subject of numerous studies, e.g., Refs. [13–28]. De Monte
[13] presented a summary of different analytical approaches that can be adopted to analyze transient heat conduction. Laplace transform method [14], Green’s function approach [15], and finite integral transform technique [16] are useful in single layer problems. However, they cannot be readily adopted to analyze multi-layered heat conduction problems [13]. In fact, quasi-orthogonal expansion technique and separation-of-variables methods are the most elegant and straightforward approaches to analyze multi-region problems [17–23]. However, such approaches can only address homogeneous or constant boundary conditions, see Refs. [24–28]. Although encountered quite often in practice, nonlinear and time-dependent boundary conditions always posed challenge to the analysis of transient heat conduction in multilayered composites. The pertinent literature has been limited to constant/simplified boundary conditions, i.e., isothermal, isoflux, or convective cooling surfaces. The solution to time-dependent heat flux or boundary temperature cases can be found by Duhamel’s theorem [14]. However, time-dependent convective cooling case, which inevitably occurs during a transient duty cycle for multilayered packages of PEEM, cannot be treated by such approach [29]; this adds extra complexity to the problem. To the best knowledge of the authors, there are only few works on transient heat conduction in multilayered composites with heat generation subjected to nonlinear or time-dependent convective–radiative cooling. Our literature review indicates that

- there is no analytical model to predict the thermal behavior of composite media with heat generation subjected to time-dependent conjugate convection–radiation.
- effects of radiation on thermal characteristics of a multilayered composite over a transient thermal duty cycle have not been considered in the existing models.
- effects of thermal contact resistance, at the interfaces between layers, have not been considered in modeling of multilayered composites under dynamic thermal load, or time-dependent boundary conditions.

To develop the present analytical model, a general time-dependent temperature is assumed for the surrounding fluid which is decomposed into simple oscillatory functions using a cosine Fourier transformation [30]. As such, conjugate radiation–convection heat transfer boundary condition is considered for the outer boundary of a composite medium. The quadratic behavior of radiation is linearized to address the boundary value problem. A quasi-orthogonal expansion technique [18] is used to find the eigenvalues of the closed-form analytical solution. Complex exponential form of the boundary condition is taken into account along with a periodic temperature response inside the media. Separation of variables method is employed and following [20] the associated discontinuous-weighting functions are found to make the obtained eigenfunctions orthogonal. Closed-form relationships are obtained for temperature distribution inside the multilayered region with arbitrary heat generations and contact resistances under time-dependent conjugate radiation–convection.

2 Governing Equations

Shown in Fig. 1 is a multilayered composite region involving $M$ parallel layers of slabs, concentric cylinders, or spheres with thermal contact resistance at the interfaces. In addition, without loss of generality, it is assumed that the surface $x_0$ represents the coordinate origin, $x_0 = 0$. In case of composite slabs, $x = 0$ is thermally insulated, while it is the symmetry line in case of concentric cylinders, and center in case of spheres. Moreover, the layer interfaces in the $x$-direction are $x_1, x_2, ..., x_j$, where $j = 1, 2, ..., M$. Let $k_j$ and $a_j$ be the thermal conductivity and the thermal diffusivity of the $j$th layer, respectively. Initially, the multilayered composite, which is confined to the domain $0 < x < x_M$, is at a uniform temperature $T_0$. It is assumed that both convection and radiation heat transfer occurs between the outer-surface boundary, $x = x_M$, that is wetted by the surrounding fluid. An arbitrary time-dependent temperature is considered for the surrounding fluid as given below:

$$T_{\text{sur}} = T_{\infty} + T_0 \gamma(t)$$

![Fig. 1 Schematic of multilayered composites in (a) Cartesian, (b) cylindrical, and (c) spherical coordinate systems](http://mechanismsrobotics.asmedigitalcollection.asme.org/ on 09/30/2015 Terms of Use: http://www.asme.org/about-asme/terms-of-use)
where $T_{\text{sur}}$ is the time-varying surrounding temperature, $T_\infty$ is a constant, $K[j]$ is the amplitude of the variation of the surrounding fluid temperature, and $\gamma(t)$ is an arbitrary function of time, respectively. Other assumptions used in the proposed unsteady heat conduction model are

- constant thermo-physical properties for all $M$ layers,
- the thickness of the multilayered composite is sufficiently thin in $x$-direction compared to the other directions, i.e., one-dimensional heat transfer.

As such, the mathematical formulation of the transient heat conduction problem herein under discussion becomes

$$
\frac{1}{\partial x^2} \frac{\partial T_j(x,t)}{\partial t} = \frac{\partial T_j(x,t)}{\partial x} + \frac{\partial (T_j(x,t))}{\partial x} = 0 \quad \text{at} \quad x = 0, t > 0
$$

$$
\frac{\partial T_j(x,t)}{\partial x} = 0 \quad \text{at} \quad x = x_j, t > 0
$$

$$
\frac{k_j}{\partial x^2} \frac{\partial T_j(x,t)}{\partial x} = k_{j+1} \frac{\partial T_{j+1}(x,t)}{\partial x} \quad \text{at} \quad x = x_{j+1}, t > 0
$$

$$
- k_M \frac{\partial T_M(x,t)}{\partial x} = h_i |_{x=x_M} \left[ T_M(x_M,t) - T_{\text{sur}}(t) \right] + \sigma_0 a x_M^2 \left( T_M(x_M,t) - T_{\text{sur}}(t) \right)
$$

$$
T_{\text{sur}} = T_\infty + T_a \times \gamma(t)
$$

where $T_{\text{sur}}$ is the instantaneous temperature of the surrounding fluid temperature, i.e., $T_0/T_{\text{sur}} > 0.75$, the forth-order effect of radiation heat transfer can be linearized, as an approximation. The validity of this assumption has been tested in various works, e.g., see Refs. [19–21]. In this paper, we consider the cases in which the ratio of initial temperature and maximum surrounding temperature is higher than 0.75, i.e., $T_0/T_{\text{sur}} > 0.75$; this justifies linearization of the radiation effects [31–33]. Therefore, Eq. (2a), time-dependent conjugate convective-radiative boundary condition, can be linearized as Eq. (4).

$$
- k_M \frac{\partial T_M(x,t)}{\partial x} = (h_i + h_o) \times \left[ T_M(x_M,t) - T_{\text{sur}}(t) \right]
$$

$$
\text{at} \quad x = x_M, t > 0
$$

where $h_i[W/m^2K]$ is the transformed radiation heat transfer coefficient defined as follows [28]:

$$
h_i = 4\sigma T_{\text{sur\_max}}^3
$$

where $T_{\text{sur\_max}}$ is the maximum surrounding fluid temperature during a transient duty cycle. It is evident that when radiation is neglected, $h_i \approx 0$, Eq. (4) reduces to time-dependent pure-convection boundary condition. Nonetheless, the dimensionless form of the energy equation, Eq. (2), becomes

$$
\frac{\partial \theta_j}{\partial \eta} = \frac{1}{\eta^2} \frac{\partial}{\partial \eta} \left( \eta^2 \frac{\partial \theta_j}{\partial \eta} \right) + g_j(\eta) \quad \text{at} \quad 0 \leq \eta \leq \frac{x_M}{x_i}, \quad F_0 > 0
$$

where $\theta_j$ is the dimensionless temperature of the $j$th layer, $F_0$ is the dimensionless time, $\eta$ is the dimensionless coordinate, and $g_j(\eta)$ is the dimensionless heat generation inside the $j$th layer. Equation (6) is subjected to the following dimensionless initial and boundary conditions:

$$
\theta_j(\eta, 0) = 0 \quad \frac{x_j-1}{x_i} \leq \eta \leq \frac{x_j}{x_i}, \quad j = 1, 2, 3, \ldots, M
$$

$$
\frac{\partial \theta_j}{\partial \eta} = A_j \left[ \theta_{j+1}(\eta = x_j/x_i, F_0) - \theta_j(\eta = x_j/x_i, F_0) \right] \quad \text{at} \quad \eta = x_j/x_i, \quad F_0 > 0
$$

$$
\frac{\partial \theta_m}{\partial \eta} = A_M \left[ \theta_1(\eta = x_M/x_i, F_0) - \theta_m(\eta = x_M/x_i, F_0) \right] + \beta_i \times \theta_m(\eta = x_M/x_i, F_0) + \Delta T_R \times \gamma(F_0)
$$

$$
\text{at} \quad \eta = x_M/x_i, \quad F_0 > 0
$$

where $A_j = \delta_j x_i / k_j$ is the dimensionless contact conductance between the $j$th and $j+1$th layers, $B_i = (h_i + h_o) x_j / k_j$ is the combined radiative-convective Biot number, and $\Delta T_R = B_i \times T_a / T_\infty - 0$ is a dimensionless number. For sake of generality, the imposed boundary condition, Eq. (6e), is considered as follows:

$$
A_m \times \frac{\partial \theta_M}{\partial \eta} (\eta = x_M/x_i, F_0) + B_{out} \times \theta_m(\eta = x_M/x_i, F_0) = C_{out} + \Delta T_R \times \gamma(F_0)
$$

$$
\text{at} \quad \eta = x_M/x_i, \quad F_0 > 0, \quad B_{out} \neq 0
$$

For a constant surrounding fluid temperature, Chapman [31] showed that when the initial temperature of the medium is close
where $A_{out}$, $B_{out}$, and $C_{out}$ are constant values. It should be noted that when $A_{out} = 1$ and $B_{out} = C_{out} = Bi$, Eq. (7) reduces to Eq. (6e). In addition, when $A_{out} = 0$, the boundary condition, Eq. (7), indicates the case in which the outer-surface boundary of a composite medium, $x = x_M$, undergoes dynamic time-dependent temperature.

3 Model development

A new model is developed to predict the thermal response of a multilayered composite in Cartesian, cylindrical, and spherical coordinates. It is assumed that the multilayered composite transfers heat via convection and radiation to the surrounding fluid whose temperature varies arbitrarily over time. Thermal contact resistances between the layers and arbitrary heat generation inside each layer are taken into consideration. The methodology is presented for: (i) a composite medium with arbitrary number of layers; The Appendix; and (ii) a composite slab (a two-die stack) consisting of two parallel layers with arbitrary heat generations and a thermal interface material (TIM) as an example in Sec. 3.1, see Fig. 2. It should be noted that the TIM layer is used to schematically represent the thermal contact resistance at the interface.

$$\tan \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right) - \frac{\sqrt{\mu_2}}{K} \tan \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right) \tan \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right)$$

$$1 + \frac{\sqrt{\mu_2}}{K} \tan \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right) - \frac{\sqrt{\mu_2}}{K} \tan \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right)$$

$$\frac{(A_{out} \lambda_n)}{\sqrt{\mu_2}} \tan \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right) - B_{out}$$

$$= \left( \frac{A_{out} \lambda_n}{\sqrt{\mu_2}} \right) + B_{out} \tan \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right)$$

Thus, with respect to Eq. (A18), the temperature distribution inside the entire medium is obtained as follows:

$$\theta_j(\eta, Fo) = \frac{C_{out}}{B_{out}} \times \cos(\alpha Fo + \varphi)$$

$$+ \sum_{n=1}^{\infty} R_{jn}(\eta) \left\{ \begin{array}{l}
A_n \cos(\alpha \varphi) \times e^{-\left(\int_{0}^{\varphi} \int_{0}^{\varphi} \right)} \\
E_n \times \left\{ \cos(\alpha Fo + \varphi) - \frac{\lambda_n}{E_n} \times \sin(\alpha Fo + \varphi) \right\} \\
+ G_n \frac{\lambda_n}{E_n} \\
\end{array} \right\}$$

The definition of the parameters used in Eq. (9) is provided in Table 1. The constants $C_{2n}$ and $D_{2n}$ are evaluated by applying Eq. (A4)

$$C_{2n} = \cos(\lambda_n) \cos \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right) + \sqrt{\mu_2} \sin(\lambda_n) \sin \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right)$$

$$- \frac{\lambda_n}{\lambda_1} \sin(\lambda_n) \cos \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right)$$

$$D_{2n} = \cos(\lambda_n) \sin \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right) - \sqrt{\mu_2} \sin(\lambda_n) \cos \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right)$$

$$- \lambda_n \sin(\lambda_n) \cos \left( \frac{\lambda_n}{\sqrt{\mu_2}} \right)$$

$$T_{in}(t) \quad \dot{q}_{out}(t) \quad \dot{q}_{con}(t)$$

**Fig. 2** Schematic of a two-die stack, and TIM subjected to time-dependent conjugate convection–radiation with a surrounding fluid

4 Numerical Study

An independent numerical simulation of the present unsteady one-dimensional heat conduction inside a two-layered slab is also conducted using the commercially available software, Comsol. Multiphysics. The thermal contact resistance between the layers is modeled by a thin thermally resistive layer. Besides, the heat generation inside the layers is taken into consideration. Furthermore, the assumptions stated in Sec. 2 are used in the numerical analysis. Harmonic and arbitrary time-dependent temperatures, Eqs. (12) and (13), are considered for the surrounding fluid. Conjugate convective–radiative heat transfer between the outer-surface boundary and the surrounding fluid are taken into account. Grid independence of the results was also verified by using three different grid sizes, namely, 1500, 2000, and 2500 cells. Since there was a little difference between the simulation results from the fine and medium-sized grids (only 1% at the most), the medium grid size was chosen for modeling to reduce the computational time. The geometrical and thermal properties used in the baseline case for numerical simulation of a two die stack are listed in Table 2. Such geometry encountered quite often in a range of practical applications including power electronics cooling, microelectronics, IGBT packages in HEVs, and high power applications, see Refs. [34] and [35]. The values associated with the TIM layer for the base-line case are in accordance with the properties of The Dow corning TC-5022 which is thermal grease extensively used in the aforementioned applications, see Ref. [34]. The convective heat transfer coefficient is also assumed, as $h_c = 18,000$/W/m²/K [36], for the base-line case since this is the average heat transfer coefficient in most electronic cooling applications, see Refs. [35] and [37]. In addition, following Ref. [35], heat flux of 200(W/cm²) is considered for the silicon chip heat generation as a typical peak value that occurs in the IGBTs. The
maximum relative difference between the analytical results and the numerical data is less than 6%; more detail is presented in Sec. 5.

5 Results and Discussion

The analytical model described in Sec. 3.1, for two parallel slabs (a two-die stack) are represented here in graphical form and compared with the numerical data obtained in Sec. 4. As such, the results are presented for: (i) a harmonic surrounding fluid temperature, and (ii) arbitrary time-dependent surrounding fluid temperature. In fact, the former case indicates a harmonic conjugate convective–radiative boundary condition, and the latter accounts for an arbitrary time-dependent conjugate convection–radiation. A code is developed using commercially available software MAPLE 15 to solve the transcendental relationship, Eq. (8). Our study shows that using the first 50 terms in the series solution, Eq. (9), is accurate enough to obtain the temperature distribution inside the media up to four decimal digits. Note that the number of series terms can be notably reduced for large time scales since for large values of \( n, T_n \rightarrow \infty \), and the exponential term in Eq. (A18) drops remarkably.

5.1 Harmonic Surrounding Fluid Temperature. In this case, the surrounding fluid temperature is considered harmonically as follows:

\[
T_{\text{sur}}(t) = T_{\infty} + T_d \cos(\omega t + \varphi) \tag{11}
\]

The dimensionless form of Eq. (11) is as given below:

\[
\theta_{\text{sur}} = \frac{T_{\text{sur}} - T_0}{T_{\infty} - T_0} = 1 + \Delta \theta_d \cos(\omega \theta_0 + \varphi) \tag{12}
\]

where \( \omega = \Omega \times \frac{x_1}{z_1} \) and \( \Delta \theta_d = T_d / T_{\infty} - T_0 \) are the dimensionless angular frequency and amplitude of the surrounding fluid temperature, respectively. As such, effects of thermal contact resistance on the thermal characteristics of the studied composite medium are shown. In addition, optimum conditions are found to maximize the amplitude of the interfacial heat flux of a composite medium under harmonic boundary condition.

5.1.1 Effects of the Thermal Contact Resistance, \( \delta_{\text{h}_{1}} \). Figure 3 shows the variations of the dimensionless temperature of the insulated axis \( \theta_{\text{h}_{1}} \), Eq. (9), against the Fourier number for different values of dimensionless conductance between the layers, \( \Lambda_1 = \delta_{\text{h}_{1}} / k_1 \). Harmonic temperature is assumed for the surrounding fluid temperature, Eq. (12), where \( T_{\infty} = 350(\text{K}), T_0 = 50(\text{K}) \), and \( \Omega = 49.2(\text{rad/s}) \). It should be noted that when \( \Lambda_1 \rightarrow \infty \), the thermal contact resistance between the layers becomes negligible, and the present model yields the solution for a two-layer slab with perfect contact at the interface. As shown in Fig. 3, there is a good agreement between the analytical results, and the obtained numerical data; maximum relative difference less than 2.7%. The following values are assumed for the dimensionless variables: \( \mu_2 = z_2 / x_1 = 1.3, K_1 = k_2 / k_1 = 2.71, \eta_2 = x_2 / x_1 = 2.33, \) and \( \alpha = 0.1 \pi \). To compare the analytical results with the numerical data, according to Ref. [34], the thermal conductivity of the resistive layer is considered as 4.2, 0.5[W/m·K], which correspond to \( \Lambda_1 = 0.135, 0.0676 \), and 0.0169, respectively.

As expected, the higher the contact conductance between the layers, the higher the amplitude of the temperature field inside the composite medium would become. The peak-shifting trend of the insulated axis temperature for different values of dimensionless interface conductance shows a “thermal lag” of the system caused by the thermal contact resistance. Irrespective of thermal contact resistance, the temperature inside the media fluctuates with the angular frequency of the surrounding temperature. In addition, as \( \Lambda_1 \rightarrow 0 \), the interface acts as a thermal insulation, and the temperature of the silicon chip is significantly less than that of high conductance at the interface.

5.1.2 Optimization of the System. When a multilayered package is exposed to a fluid with harmonic temperature, there is an optimum value for the angular frequency at which the amplitude of the interfacial heat flux is maximized. This optimum angular frequency maximizes the heat flux for given values of the dimensionless parameters \( K_1, \mu_2, \) and \( \eta_2 \). Figure 4 shows the variations of the maximum interfacial heat flux versus the angular frequency for different values of the thickness ratio. The thermal contact resistance between the layers, and the heat generation inside the layers are neglected. Here the values of the thermal conductivity and diffusivity ratios are \( K_1 = k_2 / k_1 = 2.71 \) and \( \mu_2 = z_2 / x_1 = 1.3 \), respectively, for the two-die stack described earlier.

As the thickness ratio increases, the maximum interfacial heat flux decreases. This is attributed to the thermal inertia of the upper layer which decreases the rate of heat transfer at the interface. For low thickness ratios, i.e., \( \eta < 2 \), as the angular frequency increases, the interfacial heat flux starts increasing to form a hump.

### Table 1 Definition of the parameters in Eq. (9)

<table>
<thead>
<tr>
<th>Parameter, ( j )</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{1n} )</td>
<td>( \rho ) (kg/m³)</td>
</tr>
<tr>
<td>( R_{2n} )</td>
<td>Thermal conductivity, (W/m/K)</td>
</tr>
<tr>
<td>( E_n )</td>
<td>Thermal capacity, (J/kg/K)</td>
</tr>
<tr>
<td>( G_n )</td>
<td>Thickness, (mm)</td>
</tr>
<tr>
<td>( A_n )</td>
<td>Heat load (W/cm²)</td>
</tr>
</tbody>
</table>

### Table 2 Values of the thermal and geometrical properties for the baseline case in the numerical simulation

<table>
<thead>
<tr>
<th>Layer, ( j )</th>
<th>Density, ( \rho ) (kg/m³)</th>
<th>Thermal conductivity, ( k ) (W/m/K)</th>
<th>Thermal capacity, ( c_p ) (J/kg/K)</th>
<th>Thickness, ( x ) (mm)</th>
<th>Heat load ( W/cm² )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (silicon chip), [37]</td>
<td>2330</td>
<td>148</td>
<td>712</td>
<td>0.75</td>
<td>200</td>
</tr>
<tr>
<td>TIM Layer (Dow Corning TC-5022), [38]</td>
<td>—</td>
<td>4</td>
<td>—</td>
<td>0.15</td>
<td>—</td>
</tr>
<tr>
<td>2 (copper heat spreader), [37]</td>
<td>8933</td>
<td>401</td>
<td>385</td>
<td>1</td>
<td>—</td>
</tr>
</tbody>
</table>

\( h_c = 18,000 \text{W/m}^2/\text{K}, T_0 = 300(\text{K}) \)
at the optimum angular frequency, $\omega \approx 0.3$. Beyond the optimum point, the interfacial heat flux decreases until it asymptotically approaches the value associated with the constant term of the surrounding fluid temperature. Moreover, variable coolant temperature over time does not necessarily decrease the performance of multilayered electronic packages during a thermal transient duty cycle. In fact, when the coolant temperature varies with the optimum angular frequency, i.e., $\omega \approx 0.3$, the amplitude of heat dissipation from a multilayered package increases compared to that of constant surrounding fluid temperature. For high thickness

Fig. 3 Variations of the dimensionless temperature of the insulated axis $\theta_{x=0}$, Eq. (9), against the Fourier number for different values of the dimensionless conductance between the layers, $\Lambda_1$

Fig. 4 Variations of the maximum interfacial heat flux, Eq. (A12), versus the angular frequency for different values of the thickness ratios

<table>
<thead>
<tr>
<th>$T_{\infty}$ = 373(K)</th>
<th>$\Omega_1$(rad/s)</th>
<th>$\Omega_2$(rad/s)</th>
<th>$\Omega_3$(rad/s)</th>
<th>$\Omega_4$(rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>20</td>
<td>5</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>$T_0$ = 300(K)</td>
<td>$\omega_1$(rad)</td>
<td>$\omega_2$(rad)</td>
<td>$\omega_3$(rad)</td>
<td>$\omega_4$(rad)</td>
</tr>
<tr>
<td>0</td>
<td>$-\pi/8$</td>
<td>$-\pi/4$</td>
<td>$-\pi/2$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 Parameters in Eq. (13)
The coolant temperature of the multilayered packages associated with the PEEMS of HEVs varies arbitrarily over time. As such, arbitrary time-dependent conjugate convective–radiative boundary condition is considered for the outer boundary of the composite medium exchanges heat with a fluid whose temperature varies arbitrarily over time as follows:

\[ T_{\text{sur}} = T_\infty + \sum_{i=1}^{4} (T_i)_0 \cos(\omega t + \phi_i) \]  

(13)

The surrounding fluid is assumed to vary arbitrarily over time as follows:

\[ T_{\text{sur}} = T_\infty + \sum_{i=1}^{4} (T_i)_0 \cos(\omega t + \phi_i) \]  

(15)

5.2 Arbitrary Time-Dependent Surrounding Temperature

The coolant temperature of the multilayered packages associated with the PEEMS of HEVs varies arbitrarily over time. Thus, a superposition technique is applied to the results of the harmonic boundary condition, Sec. 5.1, using the cosine Fourier transformation [30]. It is intended to obtain the temperature distribution inside a two-die stack exchanging heat with a fluid whose temperature varies arbitrarily over time. Following Ref. [34], the surrounding temperature is assumed to fluctuates around 100 °C. The analytical results are compared with the numerical simulation data obtained in Sec. 4. The results are shown here are dimensional; we believe this will provide a better sense for the problem herein under consideration. The temperature of the surrounding fluid is assumed to vary arbitrarily over time as follows:

\[ T_{\text{sur}} = T_\infty + \sum_{i=1}^{4} (T_i)_0 \cos(\omega t + \phi_i) \]  

(13)

The values of the parameters in Eq. (13) are presented in Table 3. The geometrical and thermophysical properties of the composite medium are previously given in Table 2. The heat generation inside each layer is considered to be as follows:

\[ q_j (\text{W/m}^3) = \begin{cases} 0.266 \times 10^6 & \text{if } j = 1 \\ 0 & \text{if } j = 2 \end{cases} \]  

(14)

6 Conclusion

A new analytical model is presented for the solution of heat diffusion inside a multilayered composite medium. It is assumed that the outer boundary of the composite medium exchanges heat with a surrounding fluid via convection and radiation mechanisms. The temperature of the surrounding fluid is assumed to vary dynamically over time. As such, arbitrary time-dependent conjugate radiation–convection boundary condition is considered for the outer boundary of the studied medium. Arbitrary heat generation inside each layer and the thermal contact resistance between the layers are taken into consideration. Analytical outlines are presented for optimal design of multilayered systems, which undergo a thermal duty cycle. The present analytical results are verified successfully with the obtained independent numerical data.

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Nomenclature

A = matrix in Eq. (A4)

\( A_{in} \) = constant Eq. (A19)

\( A_{out} \) = constant in Eq. (7)

\( B_{in} \) = constant in Eq. (7)

\( B_{out} \) = constant in Eq. (7)

\( c_p \) = specific heat (J/kg/K)

\( c_m \) = integration coefficient, Eq. (A3)

\( C_{in} \) = constant in Eq. (7)

\( D_{in} \) = integration coefficient Eq. (A3)

\( E_{in} \) = constant Eq. (A14)

\( F \) = constant Eq. (A7)

\( F_0 \) = Fourier number = \( x_t / x_j^2 \)

\( g_j \) = dimensionless heat generation inside the jth layer

\( G_j \) = constant Eq. (A15)

\( H \) = constant Eq. (A10)

\( h_c \) = convective heat transfer coefficient (W/m²/K)

\( h_r \) = radiative heat transfer coefficient (W/m²/K)

\( i \) = complex variable = \( \sqrt{-1} \)

\( J_0 \) = zero-order Bessel function of the first kind

\( J_1 \) = first-order Bessel function of the first kind

\( k_j \) = thermal conductivity of the jth layer (W/m/k)

\( k_j = k_{j+1} / k_M \) = number of layers

\( p \) = integer number (\( p = 0, 1, 2 \) for slabs, cylinders and spheres, respectively.)

\( P \) = matrix in Eq. (A5a) \( \hat{q}_j \) = heat generation inside the jth layer (W/m³)

\( q_{in}^* \) = dimensionless interfacial heat flux at the interface of the jth layer (W/m²)

\( R_m \) = mth eigenfunction associated with \( \lambda_n \) for the jth layer

\( R_{out} \) = \( \partial R_j / \partial \eta \) at \( \eta = x_{M} / x_j \)

\( t \) = time (s)

\( T_0 \) = amplitude of surrounding fluid temperature (K) Eq. (1)

\( T_j \) = temperature of the jth layer of the composite media (K)

\( T_0 \) = initial temperature of the system (K)

\( T_{\infty} \) = constant term of surrounding fluid temperature Eq. (1)

\( w \) = weight function Eq. (A8)

\( x \) = space coordinate (either rectangular, cylindrical or spherical)

\( x_j \) = values of space coordinate at the inner boundary surfaces

\( x_{in} \) = space coordinate at the outer boundary surface

\( x_0 \) = coordinate origin

\( Y_0 \) = zero-order Bessel function of the second kind

\( Y_1 \) = first-order Bessel function of the second kind

\( \Delta T_{in} \) = dimensionless amplitude of the imposed temperature

\( \Delta T_R \) = dimensionless number Eq. (6e)

Greek Symbols

\( \alpha_j \) = thermal diffusivity of the jth layer (m²/s)

\( \Gamma \) = function of \( F_0 \) number Eq. (A12)

\( \gamma(t) \) = arbitrary function of time Eq. (1)

\( \delta_j \) = contact conductance between the jth and j + 1th layers (W/m²/K)

\( \varepsilon \) = outer surface emisivity

\( \eta \) = dimensionless coordinate = \( x / x_1 \)

\( \eta_j \) = dimensionless coordinate at the boundaries = \( x_j / x_1 \)

\( \theta \) = dimensionless temperature = \( (T - T_0) / (T_{\infty} - T_0) \)

\( \theta_{in} \) = dimensionless temperature of the jth layer = \( (T_j - T_0) / (T_{\infty} - T_0) \)

\( \lambda \) = separation constant

\( \mu_j \) = dimensionless thermal diffusivity ratio

\( \sigma \) = Estephan–Boltzmann constant = \( 5.669 \times 10^{-8} \) (W/m²/K⁴)

\( \varphi \) = phase shift, Eq. (A12)

\( \phi_{in} \) = solution to Eq. (A2) corresponding to \( \lambda_n \)

\( \psi_{in} \) = solution to Eq. (A2) corresponding to \( \lambda_n \)

\( \Omega \) = angular frequency (rad/s)

\( \omega \) = dimensionless angular frequency

\( \omega \) = dimensionless angular frequency of the surrounding fluid temperature

\( \omega \) = \( \Omega x_j^2 / \lambda_j \) \( \Lambda_j \) = dimensionless conductance between the layers \( \Lambda_j = \delta_j x_j / k_j \)

Subscripts

\( c \) = convective

\( j \) = jth layer defined in the domain

\( j = 1, 2, ..., M \)

\( n \) = integer number positive

\( 0 \) = initial condition

\( r \) = radiative

Abbreviations

Cond. = conduction

Conv. = convection

IGBT = integrated gate bipolar transistor

Max. = maximum

Rad. = radiation

Sur. = surrounding

TIM = thermal interface material

Appendix

Considering Eq. (7), any type of time-dependent surrounding temperature, \( \gamma(t) \), can be decomposed into simple cosine functions using a cosine Fourier transformation [30]. As such, we develop the present model for a harmonic surrounding temperature, and the results can be generalized to cover the cases with arbitrary time variations in surrounding temperature by using a superposition. As such, the following expression is considered as the boundary condition on the outer surface of a composite medium.

\[
A_{out} \frac{\partial \theta_M(\eta = x_M / x_j, F_0)}{\partial \eta} + B_{out} \frac{\partial \theta_M(\eta = x_M / x_j, F_0)}{\partial \eta} = C_{out} \cos(\omega F_0 + \varphi) \quad \text{at} \quad \eta = \frac{x_M}{x_j}, F_0 > 0, \quad B_{out} \neq 0
\] (A1)

where \( A_{out}, B_{out}, \) and \( C_{out} \) are constant values; besides, \( \varphi \) is the phase shift. Therefore, when \( \omega = 0, \) Eq. (A1) addresses constant convective boundary conditions, which is the first term on the right-hand side of Eq. (7). When \( \omega \neq 0, \) Eq. (A1) can be used to find the solution of Eq. (7) for the cosine Fourier transformation of \( \gamma(t) \) by a superposition technique.

A.1. Solution to M-Layer Composite Medium. The system of eigenvalue problems associated with Eq. (6) is given below:

\[
\mu_j \frac{1}{\eta_j^2} \frac{d}{d \eta} \left( \eta_j^2 \frac{d R_j}{d \eta} \right) = -\lambda_j^2 R_j \quad j = 1, 2, 3, ..., M
\] (A2)

With the following homogeneous boundary conditions:

\[
\frac{\partial R_1}{\partial \eta} = 0 \quad \text{at} \quad \eta = 0
\] (A2a)

\[
\frac{\partial R_j}{\partial \eta} = \Lambda_j [R_{j+1}(\eta = x_j / x_1) - R_j(\eta = x_j / x_1)]
\] (A2b)

\[
\frac{\partial R_j(\eta = x_j / x_1)}{\partial \eta} = K_j \frac{\partial R_{j+1}(\eta = x_j / x_1)}{\partial \eta}
\] (A2c)

\[
\eta = j = 1, 2, 3, ..., (M - 1)
\]
where $R_{jn}$ is the eigenfunction in the $j$th layer associated with the $n$th eigenvalue, $\lambda_n$. The general solution of the above eigenvalue problem is in the form of

$$ R_{jn}(\eta) = \begin{cases} R_{1n}(\eta) = \phi_{jn}(\eta) & \eta \leq 0 \\ R_{jn}(\eta) = C_{jn}\phi_{jn}(\eta) + D_{jn}\psi_{jn}(\eta) & \eta \leq \frac{S_j}{x_1} \\ C_{jn} & \frac{S_j}{x_1} \leq \eta \leq \frac{S_j}{x_1} \\ D_{jn} & \frac{S_j}{x_1} \leq \eta \leq \frac{S_j}{x_1} \\ j = 2, 3, \ldots, M \end{cases} \quad (A3) $$

where the functions $\phi_{jn}(\eta)$ and $\psi_{jn}(\eta)$ are two linearly independent solutions of Eq. (A2). A list of the function $\phi_{jn}(\eta)$ and $\psi_{jn}(\eta)$ for slabs, cylinders, and spheres can be found in Ref. [38]. $\phi_{jn}(\eta)$ function is excluded from the solution, Eq. (A3), because of the boundary condition at the origin, see Eq. (6b). The remaining boundary conditions, Eq. A2(a–d) yields

$$ Av = 0 \quad (A4) $$

where $v = [1, C_{2n}, D_{2n}, C_{3n}, D_{3n}, \ldots, C_{Mn}, D_{Mn}]^T$, and the matrix $A$ as well as its components are defined in Appendices A and B of our previous work [38]. However, $P_M$ is defined as follows:

$$ P_M = [A_{out}\phi_{Mn} + B_{out}\phi_{Mn} A_{out}\psi_{Mn} + B_{out}\psi_{Mn}]_{\eta = -\infty} \quad (A5a) $$

Equation (A4) yields $2M - 1$ homogeneous simultaneous equations for $y_j, j = 1, 2, 3, \ldots, M$. Nontrivial solution exist if the determinant of the coefficients is zero,

$$ \det A = 0 \quad (A6) $$

Equation (A6) can be solved for the eigenvalues $\lambda_n$, see Ref. [38] for more detail. Rearranging Eq. (A2) the “discontinuous-weighting function” can be found by the method introduced by Yeh [19], see Ref. [38].

$$ \frac{d}{d\eta} \left( F_{jn}^\eta \frac{dR_{jn}}{d\eta} \right) + \lambda_n^2 F_{jn}^\eta R_{jn} = 0 \quad (A7) $$

Note that $F_j$ is a constant within the interval $x_{j-1}/x_1 \leq \eta \leq x_j/x_1$, yet unknown. Based on Refs. [19] and [22], the constants $F_j$ should be determined such that the functions $R_{jn}$ become orthogonal with respect to the discontinuous-weighting function $w(\eta)$. As such, the orthogonality factors are given below:

$$ w(\eta) = \frac{\eta}{\mu_j} F_j \frac{x_j - x_{j-1}}{x_1}, \quad j = 1, 2, \ldots, M \quad (A8) $$

Following Yeh [19], the following relationship is developed to evaluate the constants $F_j$, to form the weighting functions:

$$ F_1 = 1, \quad F_j = \prod_{j-1} K_{j-1} \quad j = 2, 3, \ldots, M \quad (A9) $$

Now that the constants $F_j$ are determined, the weighting function $w(\eta)$ is known. Therefore, any function $I(\eta)$ can be expanded inside the entire multilayered medium as follows:

$$ I(\eta) = \sum_{n=1}^{\infty} H_n R_{jn}(\eta) \quad 0 \leq \eta \leq \frac{S_j}{x_1} \quad (A10) $$

The expansion is carried out over the range of $0 \leq \eta \leq \frac{S_j}{x_1}$, spanning all $M$ layers. The unknown coefficients $H_n$ in Eq. (A10) are determined by a generalized Fourier analysis over the entire range of $M$ layers and is given in the following form:

$$ H_n = \int_{j=1}^{M} I(\eta)w(\eta)R_{jn}(\eta)d\eta \quad (A11) $$

Using a separation-of-variables method, the temperature distribution inside the entire media is considered in the following form:

$$ \theta_j(\eta, Fo) = \frac{C_{out}}{B_{out}} e^{(i\omega + j^2\eta)} \frac{\sum_{n=1}^{\infty} R_{jn}(\eta)\Gamma(Fo)e^{(i\omega + j^2\eta)}}{\sum_{j=1}^{M} \int_{layerj} w(\eta)[R_{jn}(\eta)]^2 d\eta} \quad j = 1, 2, \ldots, M \quad (A12) $$

where $i = \sqrt{-1}$. Note that for convenience, the complex exponential function is assumed for the temperature distribution. Clearly, the final solution is the real part of the sought-after solution. Substituting Eq. (A12) into Eq. (6), and expanding the angular frequency over the entire medium, after some algebraic manipulation one obtains:

$$ \sum_{n=1}^{\infty} R_{jn}[\Gamma(Fo) + (i\omega + j^2\eta)\Gamma(Fo) + iE_n - G_n \times e^{-i\omega}}, e^{-i\omega + j^2\eta}] \times 0 \quad (A13) $$

where, based on Eq. (A11), the coefficients $E_n$ and $G_n$ can be determined by the following relationships:

$$ E_n = \frac{C_{out}}{B_{out}} \frac{\sum_{j=1}^{M} \int_{layerj} w(\eta)R_{jn}(\eta)d\eta}{\sum_{j=1}^{M} \int_{layerj} w(\eta)[R_{jn}(\eta)]^2 d\eta} \quad (A14) $$

$$ G_n = \frac{\sum_{j=1}^{M} \int_{layerj} \xi_j(\eta)w(\eta)R_{jn}(\eta)d\eta}{\sum_{j=1}^{M} \int_{layerj} w(\eta)[R_{jn}(\eta)]^2 d\eta} \quad (A15) $$

Since in general $R_{jn}$ is not zero, we must have

$$ \Gamma(Fo) + (i\omega + j^2\eta)\Gamma(Fo) + iE_n - G_n \times e^{-i\omega)}, e^{-i\omega + j^2\eta}] = 0 \quad (A16) $$

The solution of Eq. (A16) is given by

$$ \Gamma(Fo) = A_n e^{(i\omega + j^2\eta)} \frac{E_n}{\xi_j + \omega^2} + G_n \times e^{-i\omega + j^2\eta} \quad (A17) $$

Substituting Eq. (A17) into Eq. (A12), and considering the real part of the solution.
\[ \theta_j(\eta, F_o) = \frac{C_{\text{out}}}{B_{\text{out}}} \times \cos(\alpha F_o + \varphi) \\
+ \sum_{n=1}^{M} R_{\text{in}}(\eta) \left\{ \frac{A_n \times \cos(\varphi) \times e^{-\left(\frac{j^2 
abla^2}{\alpha^2}\right)}}{\alpha^2} \right\} \left\{ E \cdot \left( \frac{\cos(\alpha F_o + \varphi) - \frac{j^2}{\alpha^2} \times \sin(\alpha F_o + \varphi)}{\omega^2 + \frac{j^2}{\alpha^2}} \right) \times \frac{G_n}{\lambda_n^2} \right\} \right\} \]

\[(A18)\]

The coefficients \(A_n\) can be obtained by using the initial condition, Eq. (6a), together with the orthogonality properties of the eigenfunctions

\[ A_n = \frac{1}{\cos(\varphi)} \times \left\{ E_n \left[ \frac{\cos(\varphi) - \frac{j^2}{\alpha^2} \times \sin(\varphi)}{\omega^2 + \frac{j^2}{\alpha^2}} - \frac{\cos(\varphi)}{\omega} \right] \frac{G_n}{\lambda_n^2} \right\} \]

\[(A19)\]

The interfacial heat flux is determined by the following relationship:

\[ q'_j = \frac{\partial \theta_j}{\partial \eta} \mid_{\eta=\frac{\eta}{\lambda}} \]

\[(A20)\]

References


