1 Graph Basics

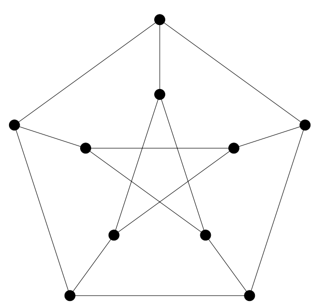
What is a graph?

Graph: a *graph* G consists of a (finite) set of vertices, denoted $V(G)$, a (finite) set of edges, denoted $E(G)$, and a relation called *incidence* so that each edge is incident with either one or two vertices, called its *ends*. For convenience, we may also denote a graph G by the ordered pair of its vertices and edges, so $G = (V, E)$ is understood to be a graph with vertex set V and edge set E.

Adjacent: Two distinct vertices u, v are *adjacent* if there is an edge with ends u, v. In this case we let uv denote such an edge.

Loops, Parallel Edges, and Simple Graphs: An edge with only one endpoint is called a *loop*. Two (or more) distinct edges with the same ends are called *parallel*. A graph is *simple* if it has no loops or parallel edges.

Drawing: It is helpful to represent graphs by drawing them so that each vertex corresponds to a distinct point, and each edge with ends u, v is realized as a curve which has endpoints corresponding to u and v (a loop with end u is realized as a curve with both endpoints corresponding to u). Below is a drawing of a famous graph.



The Petersen Graph

Standard Graphs

Null graph	the (unique) graph with no vertices or edges.
Complete graph K_n	a simple graph on n vertices every two of which are adjacent.
Path P_n	a graph whose vertices may be numbered $\{v_1, \ldots, v_n\}$ and edges numbered $\{e_1, \ldots, e_{n-1}\}$ so that every e_i has ends v_i and v_{i+1}. The *ends* of P_n are v_1, v_n and it has *length* $n-1$.
Cycle C_n	a graph whose vertices may be numbered $\{v_1, \ldots, v_n\}$ and edges numbered $\{e_1, \ldots, e_n\}$ so that every e_i has endpoints v_i and v_{i+1} (*modulo* n). The *length* of the cycle is n.

Deletion: If G is a graph and $S \subseteq E(G)$, we let $G - S$ denote the graph obtained from G by deleting every edge in S. Similarly, if $X \subseteq V(G)$, we let $G - X$ denote the graph obtained from G by deleting every vertex in X and any edge incident with such a vertex. If $e \in E(G)$, $v \in V(G)$, we let $G - e = G - \{e\}$ and $G - v = G - \{v\}$.

Subgraph: If G is a graph then any graph H obtained from G by deleting vertices or edges is called a *subgraph* of G and we denote this by $H \subseteq G$. If $H_1, H_2 \subseteq G$ we let $H_1 \cup H_2$ denote the subgraph of G with vertices $V(H_1) \cup V(H_2)$ and edges $E(H_1) \cup E(H_2)$. We define $H_1 \cap H_2$ analogously. A *path* (*cycle*) of G is a subgraph of G which is a path (cycle).

Degree: The *degree* of a vertex v, denoted $deg(v)$, is the number of edges incident with v where loops are counted twice.

Theorem 1.1 *For every graph* G

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|$$

Proof: Every edge contributes exactly two to the sum of the degrees. □

Corollary 1.2 *Every graph has an even number of vertices of odd degree.*

Isomorphic: We say that two graphs G and H are *isomorphic* if there exist bijections between $V(G)$ and $V(H)$ and between $E(G)$ and $E(H)$ which preserve the incidence relation. Informally, we may think of G and H as isomorphic if one can be turned into the other by renaming vertices and edges.

Connectivity

Walk: A *walk* W in a graph G is a sequence $v_0, e_1, v_1, \ldots, e_n v_n$ so that $v_0, \ldots, v_n \in V(G)$, $e_1, \ldots, e_n \in E(G)$, and every e_i has ends v_{i-1} and v_i. We say that W is a walk *from* v_0 *to* v_n with *length* n. If $v_0 = v_n$ we say the W is *closed*, and if $v_0, \ldots, v_n$ are distinct, then (abusing notation) we call this walk a *path*.

Connected: A graph G is *connected* if for every $u, v \in V(G)$ there is a walk from u to v.

Proposition 1.3 *If there is a walk from u to v, then there is a path from u to v.*

Proof: Choose a walk W from u to v, say $u = v_0, e_1, v_1, \ldots, e_n, v_n = v$ of minimum length. Suppose (for a contradiction) that $v_i = v_j$ for some $0 \leq i < j \leq n$. Then $v_0, e_1, \ldots, v_i, e_{j+1}, v_{j+1}, \ldots, v_n$ is a walk from u to v of shorter length than W, which is contradictory. It follows that W has no repeated vertices, so it is a path. $\square$

Proposition 1.4 *A graph G is connected if and only if there is no partition $\{X, Y\}$ of $V(G)$ so that no edge has one end in X and one end in Y.*

Proof: For the 'if" direction, suppose that such a partition $\{X, Y\}$ exists. Then any walk started at a vertex in X must have all vertices in X, so there will be no walk from a vertex in X to a vertex in Y and G is not connected.

For the "only if" direction, since G is not connected we may choose $u, v \in V(G)$ so that there is no walk from u to v. Now define $X \subseteq V(G)$ to be the set of all vertices w so that there exists a walk from u to w, and let $Y = V(G) \setminus X$. Now $u \in X$ and $v \in Y$ so $\{X, Y\}$ is a partition of $V(G)$. Let us suppose (for a contradiction that there is an edge xy with $x \in X$ and $y \in Y$. Then by assumption there is a walk W from u to x. However then extending this walk by the edge xy and the vertex then gives a walk from u to y which is contradictory. Thus, no edge has one end in X and another in Y and $\{X, Y\}$ satisfies the theorem. $\square$

Proposition 1.5 *If $H_1, H_2 \subseteq G$ are connected and $H_1 \cap H_2 \neq \emptyset$, then $H_1 \cup H_2$ is connected.*

Proof: Let $u, v \in V(H_1 \cup H_2)$ and choose $p \in V(H_1) \cap V(H_2)$. Now, there exists a walk W_1 from u to p in $H_1 \cup H_2$ (in H_1 if $u \in V(H_1)$ and otherwise in H_2). Similarly, there is a walk W_2 from p to v. By concatenating W_1 and W_2, we obtain a walk from u to v. Since u, v were arbitrary vertices, it follows that G is connected. $\square$

Component: A *component* of G is a maximal nonempty connected subgraph of G. We let *comp*(G) denote the number of components of G.

Theorem 1.6 *Every vertex and every edge is in a unique component of G.*

Proof: Every vertex v is in a connected subgraph (consisting of only that vertex with no edges), so v must be contained in at least one component. However, by the previous proposition, no two components can share a vertex. It follows that every vertex is in exactly one, as required. A similar argument holds for edges. $\square$

Cut-edge: An edge e is called a *cut-edge* if there is no cycle containing e.

Proposition 1.7 *If G is a graph and $e \in E(G)$, exactly one of the following holds:*

(i) *e is a cut-edge and $comp(G - e) = comp(G) + 1$.*

(ii) *e is not a cut-edge and $comp(G - e) = comp(G)$.*

Proof: Let u, v be then ends of e (so $u = v$ if e is a loop). Now G has a cycle containing e if and only if $G - e$ contains a path from u to v which holds if and only if u, v are in the same component of $G - e$. If u, v are in the same component H of $G - e$ then $H + e$ is a component of G so $comp(G - e) = comp(G)$. If u, v are in distinct components, say H_1, H_2 of $G - e$ then $H_1 \cup H_2 + e$ is a component of G so $comp(G - e) = comp(G) + 1$. $\square$

Bipartite, Eulerian, Hamiltonian, and Turán's Theorem

Bipartite: A *bipartition* of a graph G is a pair (A, B) of disjoint subsets of $V(G)$ with $A \cup B = V(G)$ so that every edge has one end in A and one end in B. We say that G is *bipartite* if it has a bipartition.

Complete Bipartite: The *complete bipartite graph* $K_{m,n}$ is a simple bipartite graph with bipartition (A, B) where $|A| = m$, $|B| = n$, and every vertex in A is adjacent to every vertex in B.

Theorem 1.8 *For every graph G, the following are equivalent.*

(i) *G is bipartite.*

(ii) *G has no cycle of odd length.*

(iii) *G has no closed walk of odd length.*

Proof: (i) $\Rightarrow$ (ii): Assume G is bipartite and let $C \subseteq G$ be a cycle. Then every other vertex of C is in A and every other vertex is in B so C must have even length.

(ii) $\Rightarrow$ (iii): We shall prove the contrapositive. Let G have a closed walk of odd length, and choose such a walk $v_0, e_1, \ldots, v_n$ of minimum length. If there exist $1 \leq i < j \leq n$ with $v_i = v_j$, then either $j - i$ is odd and $v_i, e_i, \ldots, v_j$ is a shorter closed walk of odd length, or $j - i$ is even and $v_0, e_1, \ldots, v_i, e_{j+1}, v_{j+1}, \ldots v_n$ is a shorter closed walk of odd length. It follows that $v_1, \ldots, v_n$ must be distinct, so $(\{v_1, \ldots, v_n\}, \{e_1, \ldots, e_n\})$ is an odd cycle.

(iii) $\Rightarrow$ (i): Let G be a graph which satisfies (iii), and assume (without loss) that G is connected. Now choose a vertex $u \in V(G)$, let $A \subseteq V(G)$ ($B \subseteq V(G)$) be the set of all vertices v so that there is a path of odd length (even length) from u to v. Suppose (for a contradiction) that there exists a vertex $w \in A \cap B$. Then there is a walk W_1 from u to w of odd length, and a walk W_2 from u to w of even length. By concatenating W_1 with the reverse of W_2 we obtain a closed walk of odd length - a contradiction. It follows that $A \cap B = \emptyset$. Since G is connected, we have $A \cup B = V(G)$. Further, it follows that (A, B) is a bipartition of G, so G satisfies (i). $\square$

Induced Subgraph: A subgraph $H \subseteq G$ is *induced* if $E(H)$ contains every edge with both ends in $V(H)$. Equivalently, H is an induced subgraph of G if and only if $H = G - (V(G) \setminus V(H))$.

Theorem 1.9 *If G is a simple graph, then G is bipartite if and only if every induced cycle of G has even length.*

Proof: The "only if" direction follows from (ii) of the previous theorem. For the "if" direction, we prove the contrapositive. So, assume that G is not bipartite. By the previous theorem, we may choose an odd cycle $C \subseteq G$ of shortest length. If C is not induced, say $e \in E(G) \setminus E(C)$ has both ends in $V(C)$, then there is a smaller odd cycle using edges in $E(C) \cup \{e\}$, which is a contradiction. Thus, C is an induced cycle of odd length, as required. $\square$

Theorem 1.10 *Every loopless graph G has a bipartite subgraph H with $|E(H)| \geq \frac{1}{2}|E(G)|$.*

Proof: Let $V = V(G)$ and choose disjoint sets $A, B \subseteq V$ with $A \cup B$ so that the number of edges of G with one end in A and one end in B is maximum. Let H be the subgraph of G with vertex set V and all edges with one end in A and one end in B. Let $v \in A$ and assume that v has s edges with other endpoint in A and t edges with other endpoint in B. If $s > t$, then moving v to B increases the number of edges with one end in A and one in B, giving us a contradiction. It follows that $deg_H(v) \geq \frac{1}{2} deg_G(v)$. Then by Theorem 1.1 we have

$$2|E(H)| = \sum_{v \in V} deg_H(v) \geq \frac{1}{2} \sum_{v \in V} deg_G(v) = |E(G)|. \qquad \square$$

Proposition 1.11 *If every vertex of G has degree ≥ 2, then G contains a cycle.*

Proof: We may assume that G is loopless (since every loop forms a cycle). Now, construct a walk $v_0, e_1, v_2, \ldots$ in a greedy manner by beginning at a vertex v_0, following an edge e_1 to a vertex v_1, then following a new edge $e_2 \neq e_1$ to a vertex v_2, and so on (maintaining the property that $e_{i+1} \neq e_i$), stopping when we first revisit a vertex. Since each vertex has degree ≥ 2 (and no loops) it is always possible to choose e_{i+1} with $e_{i+1} \neq e_i$. Since $V(G)$ is finite, at some point we have $v_i = v_j$ for some $i < j$ and our procedure terminates. Now $v_i, e_{i+1}, \ldots, v_j$ forms a cycle. $\square$

Decomposition: A *decomposition* of a graph G is a list of subgraphs $H_1, \ldots, H_k$ so that $E(H_i) \cap E(H_j) = \emptyset$ whenever $1 \leq i < j \leq k$ and $\cup_{i=1}^{k} E(H_i) = E(G)$. If every H_i is a cycle, path, etc., we say that $H_1, \ldots, H_k$ is *a decomposition of G into* cycles, paths, etc.

Proposition 1.12 *If G is a graph in which every vertex has even degree, then G has a decomposition into cycles.*

Proof: Choose a maximal list of cycles $C_1, \ldots, C_m$ which are pairwise edge-disjoint. Suppose (for a contradiction) that $G - (\cup_{i=1}^{m} E(C_i))$ has at least one edge and choose a component H of this graph with $E(H) \neq \emptyset$. Now, every vertex in H has even degree, and non-zero degree, so by the previous proposition, H contains a cycle C. This contradicts our choice, thus completing the proof. $\square$

Eulerian: A closed walk of a graph G is called *Eulerian* if it uses every edge exactly once. A graph is *Eulerian* if it has an Eulerian walk.

Theorem 1.13 *A connected graph is Eulerian if and only if every vertex has even degree.*

Proof: The "only if" direction is immediate. For the "if" direction, assume that every vertex of G has even degree, and choose a closed walk $v_0, e_1, \ldots, v_n$ of maximum length subject to the constraint that each edge is used at most once. Let $S = \{e_1, \ldots, e_n\}$, and let $X = \{v_0, \ldots, v_n\}$. Suppose (for a contradiction) that $S \neq E(G)$. We claim that there must exist an edge $e \in E(G) \setminus S$ with at least one end in X. This is certainly true if $X = V(G)$. If $X \subset V(G)$, then since G is connected, there must be an edge e with one end in X and one in $V(G) \setminus X$, so $e \in E(G) \setminus S$ has one end in X. Now, let $v_i \in X$ be an end of e and consider the component H of $G - S$ which contains e and v_i. It follows from our assumptions that every vertex of H has even degree, so by the previous proposition, we may choose a cycle C of H which contains the edge e (and thus the vertex v_i). Now we may extend the walk W by taking the initial part of this walk from v_0 to v_i, then traversing the cycle C once from v_i back to itself, and then taking the final part of W from v_i to v_n. This new walk contradicts our original choice. Thus $S = E(G)$, and W is an Eulerian walk. □

Hamiltonian: A cycle C of a graph G is *Hamiltonian* if $V(C) = V(G)$. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

Observation 1.14 *Let G be a graph and let $X \subseteq V(G)$. If $|X| < comp(G - X)$, then G is not Hamiltonian.*

Proof: We prove the contrapositive. If $C \subseteq G$ is a Hamiltonian cycle, then

$$|X| \geq comp(C - X) \geq comp(G - X). \qquad \square$$

Theorem 1.15 *Let G be a simple graph with $n \geq 3$ vertices. If $deg(u) + deg(v) \geq n$ for every two non-adjacent vertices u, v, then G is Hamiltonian.*

Proof: We proceed by induction on $t = \binom{n}{2} - |E(G)|$. If $t = 0$, then G is complete, so it has a Hamiltonian cycle. Thus we may assume that $t > 0$ and we may choose two distinct non-adjacent vertices u, v. Now, add a new edge with e with ends u, v to form the graph G'. By induction, G' has a Hamiltonian cycle. If this cycle does not use e, then it is also a Hamiltonian cycle of G, so we are done. Thus, we may assume that this cycle uses e.

Therefore, we may number $V(G)$ as $v = v_1, v_2, \ldots, v_n = u$ so that v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$. Set

$$\begin{aligned} P &= \{v_i : i \geq 2 \text{ and } v_i \text{ is adjacent to } v_1\} \\ Q &= \{v_i : i \geq 2 \text{ and } v_{i-1} \text{ is adjacent to } v_n\} \end{aligned}$$

Then $|P| + |Q| = deg(v) + deg(u) \geq n$ and since $P \cup Q \subseteq \{v_2, \ldots, v_n\}$, it follows that there exists $2 \leq i \leq n$ with $v_i \in P \cap Q$, so there is an edge e with ends v_1 and v_i and an edge e' with ends v_n and v_{i-1}. Using these two edges, we may form a Hamiltonian cycle in G as desired. □

Clique: A set of vertices $X \subseteq V(G)$ is a *clique* if every pair of points in X are adjacent.

Theorem 1.16 (Turán) *If $G = (V, E)$ is a simple graph with $|V| = n$ and $|E| = m$ which has no clique on p vertices, then*

$$m \leq \left(1 - \frac{1}{p-1}\right) \frac{n^2}{2}.$$

Proof: We proceed by induction on n. As a base, note that the formula is trivial when $n = 1$. For the inductive step, we may assume that G is a maximal graph on $n > 1$ vertex which does not contain a K_p subgraph. By this assumption, we may choose a $p-1$ vertex clique $A \subseteq V$. Set $B = V \setminus A$.

Now, there are $\binom{p-1}{2}$ edges with both ends in A. Every vertex in B is adjacent to at most $p-2$ points in A, so the number of edges with one end in A and the other in B is at most $(p-2)(n-p+1)$. Finally, by induction, the graph $G - A$ has at most $\frac{1}{2}(1 - \frac{1}{p-1})(n-p+1)^2$ edges. Altogether, this gives

$$\begin{aligned} m &\leq \binom{p-1}{2} + (p-2)(n-p+1) + \frac{1}{2}\left(1 - \frac{1}{p-1}\right)(n-p+1)^2 \\ &= \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2} \end{aligned}$$

as desired. □