4 Connectivity

2-connectivity

Separation: A *separation of G of order k* is a pair of subgraphs (H_1, H_2) so that

- $H_1 \cup H_2 = G$
- $E(H_1) \cap E(H_2) = \emptyset$
- $|V(H_1) \cap V(H_2)| = k$

Such a separation is *proper* if $V(H_1) \setminus V(H_2)$ and $V(H_2) \setminus V(H_1)$ are nonempty.

Observation 4.1 *G has a proper separation of order 0 if and only if G is disconnected.*

Cut-vertex: A vertex v is a *cut-vertex* if $comp(G - v) > comp(G)$.

Observation 4.2 *If G is connected, then v is a cut-vertex of G if and only if there exists a proper 1-separation (H_1, H_2) of G with $V(H_1) \cap V(H_2) = \{v\}$.*

Proposition 4.3 *Let e, f be distinct non-loop edges of the graph G. Then exactly one of the following holds:*

(i) *There exists a cycle C with $e, f \in E(C)$*

(ii) *There is a separation (H_1, H_2) of order ≤ 1 with $e \in E(H_1)$ and $f \in E(H_2)$.*

Proof: It is clear that (i) and (ii) are mutually exclusive, so it suffices to show that (i) or (ii) holds. For this, we may assume that G is connected, and set k to be the length of the shortest walk containing e, f. We proceed by induction k. For the base case, if $k = 2$, then we may assume $e = uv$ and $f = vw$. If u, w are in the same component of the graph $G - v$, then (i) holds. Otherwise, v is a cut-vertex and (ii) holds.

For the inductive step, we may assume $k \geq 3$. Let $f = uv$ and choose an edge $f' = vw$ so that there is a walk containing e, f' of length $k - 1$. First suppose that there is a cycle C containing e, f'. If $C - v$ and u are in distinct components of $G - v$, then v is a cut-vertex and (ii) holds. Otherwise, we may choose a path $P \subseteq G - v$ from u to a vertex of $V(C) \setminus \{v\}$. Now $P \cup C + f$ has a cycle which contains e, f, so (i) holds. If there is no cycle containing e, f', then it follows from our inductive hypothesis that there is a 1-separation (H_1, H_2) with $e \in E(H_1)$ and $f' \in E(H_2)$. Suppose (for a contradiction) that $f \in E(H_1)$. Then $V(H_1) \cap V(H_2) = \{v\}$, the shortest walk containing e, f has length k and the shortest walk containing e, f' has length $< k$ which is contradictory. Thus, $f \in E(H_2)$ and (H_1, H_2) satisfy (ii). This completes the proof. □

2-connected: A graph G is 2-connected if it is connected, $v(G) \geq 3$, and G has no cut-vertex.

Theorem 4.4 *Let $G = (V, E)$ be a loopless graph with at least three vertices. Then the following are equivalent:*

(i) *G is 2-connected*

(ii) *For all $x, y \in V$ there exists a cycle containing x, y.*

(iii) *G has no isolated vertex, and for all $e, f \in E$ there is a cycle containing e, f.*

Proof: We will show (i) ⇒ (iii) ⇒ (ii) ⇒ (i). The first implication follows immediately from Proposition 4.3. For (iii) ⇒ (ii), let G satisfy (iii) and let $x, y \in V$. To find a cycle containing x, y, choose $e \in E$ incident with x and $f \in E$ incident with y and apply (iii) to e, f. Finally, for (ii) ⇒ (i), we prove the contrapositive. Choose a cut-vertex z and then choose x, y from distinct components of $G - z$. Now G has no cycle containing x, y. □

Block: A *block* of G is a maximal connected subgraph $H \subseteq G$ so that H does not have a cut-vertex. Note that if H is a block, then either H is 2-connected, or $|V(H)| \leq 2$.

Proposition 4.5 *If H_1, H_2 are distinct blocks in G, then $|V(H_1) \cap V(H_2)| \leq 1$. Furthermore, if*

Proof: Suppose (for a contradiction) that $|V(H_1) \cap V(H_2)| \geq 2$. Let $H' = H_1 \cup H_2$, let $x \in H'$ and consider $H' - x$. By assumption, $H_1 - x$ is connected, and $H_2 - x$ is connected. Since these graphs share a vertex, $H' - x = (H_1 - x) \cup (H_2 - x)$ is connected. Thus, H' has no cut-vertex. This contradicts the maximality of H_1, thus completing the proof. □

Block-Cutpoint graph: If G is a graph, the *block-cutpoint* graph of G, denoted $BC(G)$ is the simple bipartite graph with bipartition (A, B) where A is the set of cut-vertices of G, and B is the set of blocks of G, and $a \in A$ and $b \in B$ adjacent if the block b contains the cut-vertex a.

Observation 4.6 *If G is connected, then $BC(G)$ is a tree.*

Proof: Let (A, B) be the bipartition of $BC(G)$ as above. It follows from the connectivity of G that $BC(G)$ is connected. If there is a cycle $C \subseteq BC(G)$, then set H to be the union of all blocks in $B \cap V(C)$. It follows that H is a 2-connected subgraph of G (as in the proof of the previous proposition). This contradicts the maximality of the blocks in $B \cap V(C)$. □

Ears: An *ear* of a graph G is a path $P \subseteq G$ of length ≥ 1 so that all internal vertices of P have degree 2 in G and so that the ends of P have degree ≥ 3 in G An *ear decomposition* of G is a decomposition of G into $C, P_1, \ldots, P_k$ so that C is a cycle of length ≥ 3, and for every $1 \leq i \leq k$, the subgraph P_i is an ear of $C \cup P_1 \cup \ldots P_i$.

Theorem 4.7 *A loopless graph G is 2-connected if and only if it has an ear decomposition.*

Proof: For the "if" direction, let $C, P_1, \ldots, P_k$ be an ear decomposition of G. We shall prove that G is 2-connected by induction on k. As a base, if $k = 0$, then $G = C$ is 2-connected. For the inductive step, we may assume that $k \geq 1$ and that $C \cup P_1 \cup \ldots P_{k-1}$ is 2-connected. It then follows easily that $G = C \cup P_1 \cup \ldots P_k$ is also 2-connected.

We prove the "only if" direction by a simple process. First, choose a cycle $C \subseteq G$ of length ≥ 3 (this is possible by Theorem 4.4). Next we choose a sequence of paths $P_1, \ldots, P_k$ as follows. If $G' = C \cup P_1 \cup \ldots P_{i-1} \neq G$, then choose an edge $e \in E(G')$ and $f \in E(G) \setminus E(G')$, and then choose a cycle $D \subseteq G$ containing e, f (again using 4.4). Finally, let P_i be the maximal path in D which contains the edge f and has all internal vertices not contained in $V(G')$. It follows that P_i is an ear of $C \cup P_1 \cup \ldots P_i$, and when this process terminates, we have an ear decomposition. □

Menger's Theorem

Theorem 4.8 *Let G be a graph, let $A, B \subseteq V(G)$ and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

(i) *There exist k pairwise (vertex) disjoint paths $P_1, \ldots, P_k$ from A to B.*

(ii) *There is a separation (H_1, H_2) of G of order $< k$ with $A \subseteq V(H_1)$ and $B \subseteq V(H_2)$.*

Proof: It is clear that (i) and (ii) are mutually exclusive, so it suffices to show that (i) or (ii) holds. We prove this by induction on $|E(G)|$. As a base, observe that the theorem holds trivially when $|E(G)| \leq 1$. For the inductive step, we may then assume $|E(G)| \geq 2$. Choose an edge e and consider the graph $G' = G - e$. If G' contains k disjoint paths from A to B, then so does G, and (i) holds. Otherwise, by induction, there is a separation (H_1, H_2) of G' of order $< k$ with $A \subseteq V(H_1)$ and $B \subseteq V(H_2)$.

Now consider the separations (H_1+e, H_2) and (H_1, H_2+e) of G. If one of these separations has order $< k$, then (ii) holds. Thus, we may assume that e has one end in $V(H_1) \setminus V(H_2)$, the other end in $V(H_2) \setminus V(H_1)$, and both (H_1+e, H_2) and (H_1, H_2+e) have order k. Choose (H_1', H_2') to be one of these two separations with $E(H_1'), E(H_2') \neq \emptyset$ (this is possible since $|E(G)| \geq 2$) and set $X = V(H_1') \cap V(H_2')$ (note that $|X| = k$). Now, we apply the theorem inductively to the graph H_1' for the sets A, X and to H_2' for the sets X, B. If there are k disjoint paths from A to X in H_1' and k disjoint paths from X to B in H_2', then (i) holds. Otherwise, by induction there is a separation of H_1' or H_2' in accordance with (ii), and it follows that (ii) is satisfied. □

Note: The above theorem implies Theorem 3.5 (König Egerváry). Simply apply the above theorem to the bipartite graph G with bipartition (A, B). Then (i) holds if and only if $\alpha'(G) \geq k$, and (ii) holds if and only if $\beta(G) < k$ (here $V(H_1) \cap V(H_2)$ is a vertex cover).

Internally Disjoint: The paths $P_1, \ldots, P_k$ are *internally disjoint* if they are pairwise vertex disjoint except for their ends.

Theorem 4.9 (Menger's Theorem) *Let u, v be distinct non-adjacent vertices of G, and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

(i) *There exist k internally disjoint paths $P_1, \ldots, P_k$ from u to v.*

(ii) *There is a separation (H_1, H_2) of G of order $< k$ with $u \in V(H_1) \setminus V(H_2)$ and $v \in V(H_2) \setminus V(H_1)$.*

Proof: Let $A = N(u)$ and $B = N(v)$ and apply the above theorem to $G - \{u, v\}$. □

k-Connected: A graph G is *k-connected* if $v(G) \geq k+1$ and $G - X$ is connected for every $X \subseteq V(G)$ with $|X| < k$. Note that this generalizes the notion of 2-connected from Section 13. Also note that 1-connected is equivalent to connected.

Corollary 4.10 *A simple graph G with $v(G) \geq k+1$ is k-connected if and only if for every $u, v \in V(G)$ there exist k internally disjoint paths from u to v.*

Line Graph: If G is a graph, the *line graph* of G, denoted $L(G)$, is the simple graph with vertex set $E(G)$, and two vertices $e, f \in E(G)$ adjacent if e, f share an endpoint in G.

Edge cut: If $X \subseteq V(G)$, we let $\delta(X) = \{xy \in E(G) : x \in X \text{ and } y \notin X\}$, and we call any set of this form an *edge cut*. If $v \in V(G)$ we let $\delta(v) = \delta(\{v\})$.

Theorem 4.11 (Menger's Theorem - edge version) *Let u, v be distinct vertices of G and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

(i) *There exist k edge disjoint paths $P_1, \ldots, P_k$ from u to v.*

(ii) *There exists $X \subseteq V(G)$ with $u \in X$ and $v \notin X$ so that $|\delta(X)| < k$.*

Proof: Apply Theorem 4.8 to the graph $L(G)$ for $\delta(u)$ and $\delta(v)$ and k. □

k-edge-connected: A graph G is k-edge-connected if $G - S$ is connected for every $S \subseteq E(G)$ with $|S| < k$.

Corollary 4.12 *A graph G is k-edge-connected if and only if for every $u, v \in V(G)$ there exist k pairwise edge disjoint paths from u to v.*

Fans and Cycles

Subdivision: If $e = uv$ is an edge of the graph G, then we *subdivide* e by removing the edge e, adding a new vertex w, and two new edges uw and wv.

Observation 4.13

1. *Subdividing an edge of a 2-connected graph yields a 2-connected graph.*

2. *Adding an edge to a k-connected graph results in a k-connected graph.*

3. *If G is k-connected and $A \subseteq V(G)$ satisfies $|A| \geq k$, then adding a new vertex to G and an edge from this vertex to each point in A results in a k-connected graph.*

Fan: Let $v \in V(G)$ and let $A \subseteq V(G) \setminus \{v\}$. A (v, A)-*fan* of *size* k is a collection of k paths $\{P_1, \ldots, P_k\}$ so that each P_i is a path from v to a point in A, and any two such paths intersect only in the vertex v.

Lemma 4.14 *If G is k-connected, $v \in V(G)$ and $A \subseteq V(G) \setminus \{v\}$ satisfies $|A| \geq k$, then G contains a (v, A)-fan of size k.*

Proof: Construct a new graph G' from G by adding a new vertex u and then adding a new edge between u and each point of A. By the above observation, G' is k-connected, and $u, v \in V(G')$ are nonadjacent, so by Menger's theorem there exist k internally disjoint paths from u to v. Removing the vertex u from each of these paths yields a (v, A)-fan of size k in G. □

Theorem 4.15 *Let G be a k-connected graph with $k \geq 2$ and let $X \subseteq V(G)$ satisfy $|X| = k$. Then there exists a cycle $C \subseteq G$ with $X \subseteq V(C)$.*

Proof: Choose a cycle $C \subseteq G$ so that $|V(C) \cap X|$ is maximum, and suppose (for a contradiction) that $X \not\subseteq V(C)$. Choose a vertex $v \in X \setminus V(C)$ and set $k' = \min\{k, |V(C)|\}$. It follows from the above lemma that G has a $(v, V(C))$-fan of size k', say $\{P_1, \ldots, P_{k'}\}$. Since $|X \cap V(C)| < k$, it follows that there exists a cycle $C' \subseteq C \cup P_1 \cup \ldots \cup P_{k'}$ so that $\{v\} \cup (X \cap V(C)) \subseteq V(C')$. This contradiction completes the proof. □