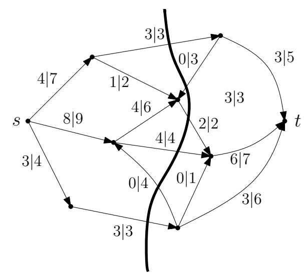
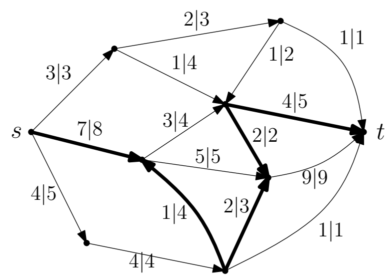
Homework 10 Solutions

Problem 1. In the following digraph, each edge is labelled as $\phi(e) \mid c(e)$ where ϕ is a (s,t)-flow and $c(e)$ is the capacity of the edge e. Either prove that there is an augmenting path from s to t of find a cut with capacity equal to the existing flow value.



Problem 2. Repeat the previous exercise for the following graph.



Problem 3. Prove or disprove: every loopless graph G has an orientation which is acyclic.

Solution: This is true. Order the vertices of G as $v_1, \dots, v_n$ and then direct each edge with ends v_i, v_j from v_i to v_j if $i < j$. It is immediate that the resulting digraph is acyclic.

Problem 4. Let D be a digraph, let $s, t \in V(D)$, and assume that $deg^+(v) = deg^-(v)$ for every vertex $v \in V(D) \setminus \{s, t\}$ and $deg^+(s) - deg^-(s) = k = deg^-(t) - deg^+(t)$. Show that there exist k edge disjoint directed paths in D from s to t.

Solution: Modify D to form a new digraph D^+ by adding k new edges directed from t to s. The graph D^+ is Eulerian, so we may choose an Euler walk of D^+. Deleting the k new edges from this walk results in k directed walks from s to t, say $W_1, \dots, W_k$. Now, from each walk W_i we may choose a directed path P_i from s to t by minimality.

Problem 5. Prove or disprove: every graph G has an orientation D so that $|\delta^+(X)|$ and $|\delta^-(X)|$ differ by at most one for every $X \subseteq V(D)$.

Solution: This is false. Let $G \cong K_4$ with $V(G) = \{a, b, c, d\}$ and let D be an orientation of G. We shall show that D does not satisfy the condition above. We may assume (without loss) that the edges incident with a are oriented (a, b), (a, c), and (d, a). Since the edge cut between $\{a, d\}$ and $\{b, c\}$ must have two edges both ways we must now have $(b, d), (c, d) \in E(D)$. The cut between $\{a, b\}$ and $\{c, d\}$ now forces $(c, b) \in E(D)$. Finally, now the cut between $\{a, c\}$ and $\{b, c\}$ is oriented poorly.

Problem 6. Prove that every connected graph G has an orientation in which the number of vertices with odd outdegree is at most 1. (Hint: choose a spanning tree T and orient the edges in $E(G) \setminus E(T)$ arbitrarily)

Solution: We shall prove that if $H \subseteq G$ is a tree, and all edges in $E(G) \setminus E(H)$ have been oriented so that every vertex $v \in V(G) \setminus V(H)$ has even outdegree, then the remaining edges of H may be oriented so that this condition is satisfied by all but at most one vertex. We prove this by induction on $|V(H)|$. As a base, note that it is trivially true if $|V(H)| = 1$. For the inductive step, choose a leaf vertex $u \in V(H)$ and let $uw \in E(H)$. Now, orient the edge uw so that u has even outdegree. By induction, we may orient the remaining edges of $H - u$ to achieve our goal. Now, we can use this result to solve the original problem by choosing a spanning tree T, orienting every edge in $E(G) \setminus E(T)$ arbitrarily, and then applying the result to T.

Problem 7. Let D be an orientation of a tree with the property that $deg^-(v) \leq 1$ for every $v \in V(D)$. Prove that there exists a vertex r so that every other vertex in the tree can be reached from r by a directed path.

Solution: Since every vertex v satisfies $deg^-(v) \leq 1$ and $\sum_{v \in V(D)} deg^-(v) = |E(D)| = |V(D)| - 1$ there is a unique vertex $r \in V(D)$ for which $deg^-(r) = 0$. Let u be any other vertex in D and consider the unique path from r to u in the underlying graph. By assumption, the first edge of this path must be directed forward out of r. It then follows that the second edge must also be directed forward (otherwise this second vertex would have two in-edges). By a straightforward induction we conclude the this is a directed path from r to u.