Homework 1 Solutions

1. Prove or disprove: If every vertex of G has degree 2, then G is a cycle.

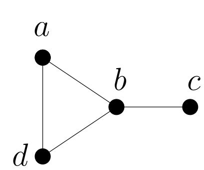
Solution: This is false. A graph with two components each of which is a cycle is a counterexample.

2. Prove that a bipartite graph has a unique bipartition (up to switching the two sets) if and only if it is connected.

Solution: First suppose that G is a disconnected bipartite graph with bipartition (A, B). Let G' be a component of G, set $X = V(G')$ and let $G'' = G - X$. Setting $A' = A \cap X$ and $B' = B \cap X$ we find that (A', B') is a bipartition of G' and similarly $A'' = A \setminus X$ and $B'' = B \setminus X$ is a bipartition of G''. Now, since there are no edges between X and the rest of the graph it follows that $(A, B) = (A' \cup A'', B' \cup B'')$ and $(A' \cup B'', B' \cup A'')$ are inequivalent bipartitions of G.

Next suppose that (A, B) and (A', B') are bipartitions of G,with $A' \neq A$ and $A' \neq B$. Set $X = (A \cap A') \cup (B \cap B')$ and $Y = (A \cap B') \cup (A' \cap B)$. It follows immediately that X, Y are disjoint. If $X = \emptyset$ then $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$ so $A \subseteq B'$ and $B' \subseteq A$. But then we have $A = B'$ and $B = A'$ giving us a contradiction. Thus $X \neq \emptyset$ and by a similar argument $Y \neq \emptyset$. It follows from the existence of our bipartitions that any edge with an end in $A \cap A'$ must have its other endin $B \cap B'$ and similarly any edge with an end in $B \cap B'$ must have its other end in $A \cap A'$. But then there are no edges from X to Y and therefore G is disconnected.

3. We say that a set of vertices X in a graph is *independent* if no edge has both of its ends in X. In the graph below, find all maximal paths and maximal independent sets.



Solution: We have labelled the vertices above. The maximal independent sets are now $\{a, c\}$, $\{d, c\}$, and $\{b\}$. The maximal paths are those with vertex sequence a, b, d, c, b, a, d, and c, b, d, a.

If $H_1, \ldots, H_m$ are subgraphs of G with $\cup_{i=1}^{k} E(H_i) = E(G)$ and $E(H_i) \cap E(H_j) = \emptyset$ for every $1 \le i < j \le k$ we say that $H_1, \ldots, H_m$ form a *decomposition* of G.

4. Find a decomposition of the Petersen graph into three pairwise isomorphic subgraphs. (Hint: it might help to find a drawing of Petersen with a 3-fold symmetry)

Solution: Here is an alternative description of the Petersen graph. Let G to be the graph with vertex set consisting of all two element subsets of $\{1,2,3,4,5\}$ and two vertices A, B adjacent if $A \cap B = \emptyset$. Then G is isomorphic to the Petersen graph. Now, for every partition P of $\{1,2,3,4\}$ into two sets of size two, there is a 6 vertex subgraph H_P of G which contains exactly those edges incident with a point in P. These three subgraphs give a decomposition of G into pairwise isomorphic graphs, as required.

5. Prove that K_n has a decomposition into three pairwise isomorphic subgraphs if and only if $n+1$ is not divisible by 3. (Hint: for the case where n is divisible by 3, split the vertices into three sets of equal size)

Solution: We break into cases dependent on the residue class of n modulo 3:

Case 1: $n \cong 2 \ (mod\ 3)$

In this case $|E(K_n)| = \binom{n}{2} = \frac{1}{2}n(n-1)$ is not divisible by 3, so there is no decomposition of K_n into three pairwise isomorphic subgraphs.

Case 2: $n \cong 0 \ (mod\ 3)$

Here we exhibit a decomposition of a complete graph on n vertices into three isomorphic subgraphs. Let X_0, X_1, X_2 be disjoint sets of size $\frac{n}{3}$, and for every $0 \le i \le 2$ let H_i be the simple graph with vertex set $X_i \cup X_{i+1}$ (working modulo 3) and two vertices adjacent if and only if at least one lies in X_i. Then H_0, H_1, H_2 form a decomposition of the complete graph on $X_0 \cup X_1 \cup X_2$ into pairwise isomorphic subgraphs as required.

Case 3: $n \cong 1 \ (mod\ 3)$

Again, we shall exhibit a decomposition of a complete graph on n vertices into three isomorphic subgraphs. Let X_0, X_1, X_2 be disjoint sets of size $\frac{n-1}{3}$ and let the subgraphs H_0, H_1, H_2 be as above. Now, let y be a new vertex, and for $0 \le i \le 2$ let H_i' be the graph obtained from H_i by adding the vertex y and all edges from X_i to y. Then H_0', H_1', H_2' form a decomposition of the complete graph on $X_0 \cup X_1 \cup X_2 \cup \{y\}$ into three pairwise isomorphic subgraphs as required.

6. Show that if K_n can be decomposed into triangles, then either $n-1$ or $n-3$ is a multiple of 6.

If K_n can be decomposed into triangles (cycles of length 3), then $|E(G)| = \binom{n}{2} = \frac{1}{2}n(n-1)$ must be a multiple of three, so either n or $n-1$ is a multiple of 3. Similarly, if K_n can be decomposed into triangles, then every vertex must have even degree (since each triangle uses an even number of edges at each vertex) so $2|n-1$. Thus, either $3|n$ and $2|n-1$ giving us $6|n-3$, or $3|n-1$ and $2|n-1$ which gives us $6|n-1$.

7. Prove that every simple connected graph with an even number of edges can be decomposed into paths of length 2. (Hint: induction).

Proof: We proceed by induction on $|E(G)|$. As a base case, note that the result holds trivially (i.e. using an empty decomposition) when $|E(G)| = 0$ For the inductive step, let G be a connected graph with $|E(G)|$ an even number > 2, and assume the statement is true for all graphs with fewer edges.

Since G is connected, and has at least two edges, it does not have two adjacent vertices of degree 1, so we may choose a vertex $y \in V(G)$ of degree ≥ 2. Let x, z be distinct vertices adjacent to y, let $G' = G - \{xy, yz\}$, and let $H_1, \ldots, H_m$ be the components of G'. Theorem 2.5 shows that $m \leq 3$, but further, every H_i must contain at least one of x, y, z.

If every H_i has an even number of edges (for $1 \leq i \leq m$), then the result follows by applying induction to each component, merging these lists of two-edge paths, and then appending the two edge path with edges xy, yz. Thus, we may assume that at least one of $H_1, \ldots, H_m$ has an odd number of edges. Since $m \leq 3$ and $|E(G)|$ is even, it follows that there are exactly two components, say H_1 and H_2, which have an odd number of edges. A quick case analysis reveals that it is always possible to add one of xy, yz to H_1 and add the other to H_2 (along with the ends of these edges) to form connected subgraphs H_1' and H_2' (respectively). Now the result follows by applying induction to H_1', H_2', and, if $m = 3$, H_3.