Homework 1 Solutions

1. Prove or disprove: If every vertex of $G$ has degree 2, then $G$ is a cycle.

Solution: This is false. A graph with two components each of which is a cycle is a counterexample.

2. Prove that a bipartite graph has a unique bipartition (up to switching the two sets) if and only if it is connected.

Solution: First suppose that $G$ is a disconnected bipartite graph with bipartition $(A, B)$. Let $G'$ be a component of $G$, set $X = V(G')$ and let $G'' = G - X$. Setting $A' = A \cap X$ and $B' = B \cap X$ we find that $(A', B')$ is a bipartition of $G'$ and similarly $A'' = A \setminus X$ and $B'' = B \setminus X$ is a bipartition of $G''$. Now, since there are no edges between $X$ and the rest of the graph it follows that $(A, B) = (A' \cup A'', B' \cup B'')$ and $(A' \cup B'', B' \cup A'')$ are inequivalent bipartitions of $G$.

Next suppose that $(A, B)$ and $(A', B')$ are bipartitions of $G$, with $A' \neq A$ and $A' \neq B$. Set $X = (A \cap A') \cup (B \cap B')$ and $Y = (A \cap B') \cup (A' \cap B)$. It follows immediately that $X, Y$ are disjoint. If $X = \emptyset$ then $A \cap A' = \emptyset$ and $B \cap B' = \emptyset$ so $A \subseteq B'$ and $B' \subseteq A$. But then we have $A = B'$ and $B = A'$ giving us a contradiction. Thus $X \neq \emptyset$ and by a similar argument $Y \neq \emptyset$. It follows from the existence of our bipartitions that any edge with an end in $A \cap A'$ must have its other end in $B \cap B'$ and similarly any edge with an end in $B \cap B'$ must have its other end in $A \cap A'$. But then there are no edges from $X$ to $Y$ and therefore $G$ is disconnected.

3. We say that a set of vertices $X$ in a graph is independent if no edge has both of its ends in $X$. In the graph below, find all maximal paths and maximal independent sets.

Solution: We have labelled the vertices above. The maximal independent sets are now $\{a, c\}$, $\{d, c\}$, and $\{b\}$. The maximal paths are those with vertex sequence $a, b, d, c, b, a, d$, and $c, b, d, a$. 
If $H_1, \ldots, H_m$ are subgraphs of $G$ with $\bigcup_{i=1}^{k} E(H_i) = E(G)$ and $E(H_i) \cap E(H_j) = \emptyset$ for every $1 \leq i < j \leq k$ we say that $H_1, \ldots, H_m$ form a decomposition of $G$.

4. Find a decomposition of the Petersen graph into three pairwise isomorphic subgraphs. (Hint: it might help to find a drawing of Petersen with a 3-fold symmetry)

Solution: Here is an alternative description of the Petersen graph. Let $G$ to be the graph with vertex set consisting of all two element subsets of $\{1, 2, 3, 4, 5\}$ and two vertices $A, B$ adjacent if $A \cap B = \emptyset$. Then $G$ is isomorphic to the Petersen graph. Now, for every partition $P$ of $\{1, 2, 3, 4\}$ into two sets of size two, there is a 6 vertex subgraph $H_P$ of $G$ which contains exactly those edges incident with a point in $P$. These three subgraphs give a decomposition of $G$ into pairwise isomorphic graphs, as required.

5. Prove that $K_n$ has a decomposition into three pairwise isomorphic subgraphs if and only if $n + 1$ is not divisible by 3. (Hint: for the case where $n$ is divisible by 3, split the vertices into three sets of equal size)

Solution: We break into cases dependent on the residue class of $n$ modulo 3:

Case 1: $n \equiv 2 \pmod{3}$

In this case $|E(K_n)| = \binom{n}{2} = \frac{1}{2}n(n-1)$ is not divisible by 3, so there is no decomposition of $K_n$ into three pairwise isomorphic subgraphs.

Case 2: $n \equiv 0 \pmod{3}$

Here we exhibit a decomposition of a complete graph on $n$ vertices into three isomorphic subgraphs. Let $X_0, X_1, X_2$ be disjoint sets of size $\frac{n}{3}$, and for every $0 \leq i \leq 2$ let $H_i$ be the simple graph with vertex set $X_i \cup X_{i+1}$ (working modulo 3) and two vertices adjacent if and only if at least one lies in $X_i$. Then $H_0, H_1, H_2$ form a decomposition of the complete graph on $X_0 \cup X_1 \cup X_2$ into pairwise isomorphic subgraphs as required.

Case 3: $n \equiv 1 \pmod{3}$

Again, we shall exhibit a decomposition of a complete graph on $n$ vertices into three isomorphic subgraphs. Let $X_0, X_1, X_2$ be disjoint sets of size $\frac{n-1}{3}$ and let the subgraphs $H_0, H_1, H_2$ be as above. Now, let $y$ be a new vertex, and for $0 \leq i \leq 2$ let $H_i'$ be the graph obtained from $H_i$ by adding the vertex $y$ and all edges from $X_i$ to $y$. Then $H_0', H_1', H_2'$ form a decomposition of the complete graph on $X_0 \cup X_1 \cup X_2 \cup \{y\}$ into three pairwise isomorphic subgraphs as required.
6. Show that if $K_n$ can be decomposed into triangles, then either $n - 1$ or $n - 3$ is a multiple of 6.

If $K_n$ can be decomposed into triangles (cycles of length 3), then $|E(G)| = \binom{n}{2} = \frac{1}{2}n(n - 1)$ must be a multiple of three, so either $n$ or $n - 1$ is a multiple of 3. Similarly, if $K_n$ can be decomposed into triangles, then every vertex must have even degree (since each triangle uses an even number of edges at each vertex) so $2|n - 1$. Thus, either $3|n$ and $2|n - 1$ giving us $6|n - 3$, or $3|n - 1$ and $2|n - 1$ which gives us $6|n - 1$.

7. Prove that every simple connected graph with an even number of edges can be decomposed into paths of length 2. (Hint: induction).

**Proof:** We proceed by induction on $|E(G)|$. As a base case, note that the result holds trivially (i.e. using an empty decomposition) when $|E(G)| = 0$ For the inductive step, let $G$ be a connected graph with $|E(G)|$ an even number $> 2$, and assume the statement is true for all graphs with fewer edges.

Since $G$ is connected, and has at least two edges, it does not have two adjacent vertices of degree 1, so we may choose a vertex $y \in V(G)$ of degree $\geq 2$. Let $x, z$ be distinct vertices adjacent to $y$, let $G' = G - \{xy, yz\}$, and let $H_1, \ldots, H_m$ be the components of $G'$. Theorem 2.5 shows that $m \leq 3$, but further, every $H_i$ must contain at least one of $x, y, z$.

If every $H_i$ has an even number of edges (for $1 \leq i \leq m$), then the result follows by applying induction to each component, merging these lists of two-edge paths, and then appending the two edge path with edges $xy, yz$. Thus, we may assume that at least one of $H_1, \ldots, H_m$ has an odd number of edges. Since $m \leq 3$ and $|E(G)|$ is even, it follows that there are exactly two components, say $H_1$ and $H_2$, which have an odd number of edges. A quick case analysis reveals that it is always possible to add one of $xy, yz$ to $H_1$ and add the other to $H_2$ (along with the ends of these edges) to form connected subgraphs $H'_1$ and $H'_2$ (respectively). Now the result follows by applying induction to $H'_1, H'_2$, and, if $m = 3$, $H_3$. 