Homework 2 Solutions

1. Determine the maximum size of an independent set in *Petersen*.

Solution: Petersen is a 3-regular graph on 15 vertices. Were it to contain an independent set X of size 5, then every edge of the graph must be incident with X, so then it would have to be bipartite. Since *Petersen* has a cycle of length 5, this is not the case. Therefore, the maximum size of an independent set is at most 4, and a simple check reveals a 4-vertex independent set.

2. Prove that a loopless graph G is bipartite if and only if for every subgraph $H \subseteq G$ the graph H has an independent set of size at least $\frac{1}{2}|V(H)|$.

Solution: First suppose that G is bipartite and let (A, B) be a bipartition of G. Then for every subgraph H we have that the sets $A' = A \cap V(H)$ and $B' = B \cap V(H)$ satisfy (A', B') a bipartition of H. Therefore, H must have an independent set of size at least $\min\{|A'|, |B'|\} \geq \frac{1}{2}|V(H)|$. Next suppose that G is not bipartite. Then G contains an odd cycle C, and C has no independent set of size at least $\frac{1}{2}|V(C)|$.

3. For every $k \geq 1$ find a simple disconnected graph G_k on $2k$ vertices with highest possible minimum degree. This should include a proof that any graph with higher minimum degree is connected.

Solution: Define G_k to be a graph with two components, each of which is isomorphic to K_k. Then G_k is a simple disconnected graph on $2k$ vertices with minimum degree $k - 1$.

Now we shall prove that every simple graph on $2k$ vertices with minimum degree at least k is connected (thus showing the above construction to be optimal). Let G be a graph on $2k$ vertices with minimum degree k and let $u, v \in V(G)$. If $uv \in E(G)$ then there is a path from u to v. Otherwise, v and u both have at least k neighbours from the set $V(G) \setminus \{u, v\}$ which has size $2k - 2$, so (by the pigeon-hole principle) there must exist a vertex w so that $u \sim w \sim v$. It follows that there is a path between any two vertices of G, so G is connected, as desired.

4. Let $u = v_1, e_1, \ldots, v_n, e_n, v_{n+1} = u$ be a closed walk in the graph G. Let S denote the set of all edges used in this walk and assume there is no cycle C of G for which $E(C) \subseteq S$. Prove that either $e_1 = e_n$ or there exists $1 \leq j \leq n$ so that $e_j = e_{j+1}$.

Solution: Choose $1 \leq k \leq n+1$ to be the smallest index for which $v_k \in \{v_1, v_2, \ldots, v_{k-1}$ (such a k must exist since $v_1 = v_{n+1}$). If $e_{k-1} \neq e_{k-2}$ then $\{e_1, \ldots, e_{k-1}\}$) is the edge set of a cycle, which contradicts our hypothesis. Therefore, $e_{k-1} = e_{k-2}$ as required.

5. Let G be a connected Eulerian graph and let $\{X, Y\}$ be a partition of $V(G)$. Show that the number of edges with one end in X and one end in Y is even. (Hint: consider the degree sums)

Solution: Let S be the set of edges with both ends in X and let T be the set of edges with one end in X and one end in Y. Now consider $\sum_{x \in X} deg(x)$. Every edge in S contributes 2 to this sum, and every edge in T contributes 1, so we have $\sum_{x \in X} deg(x) = 2|S| + |T|$. However, every term in the sum on the left hand side of this equation is even, so we must have $|T|$ even, as desired.

6. Let G be a loopless graph with every vertex of degree ≥ 3. Prove that G has a cycle of even length. (Hint: consider a path of maximum length)

Solution: If G has parallel edges, then it has a cycle of length two, and there is nothing left to prove. Thus, we may assume G is simple. Let P be a path in G of maximum length given by the walk $v_0, e_1, v_1, \ldots, v_n$. Now, $deg(v_0) \geq 3$ so there exist $2 \leq i < j \leq n$ with $v_0v_i, v_0, v_j \in E(G)$. If i is odd, then the unique cycle in $P + v_0v_i$ is even, so we are done. Similarly, if j is odd then $P + v_0v_j$ contains an even cycle. Thus, we may assume that both i and j are even. But then the unique cycle in $P - v_0v_1 + v_0v_i + v_0v_j$ is even.

7. Let G be a connected graph with $v(G) \geq 3$. Prove that G has two vertices x, y so that $G - \{x, y\}$ is connected, and further x and y are either adjacent or they have a common neighbor. (Hint: consider a path of maximum length)

Solution: Let $P \subseteq G$ be a path of maximum length. Since G must have a vertex of degree > 1, we have $v(P) \geq 3$. Let x be an end of P, let $y \in V(P)$ be the vertex adjacent to x in P, and let $H_0, H_1, \ldots, H_m$ be the components of $G - \{x, y\}$ where H_0 is the component containing the subgraph $P - \{x, y\}$. If $m = 0$, then $G - \{x, y\}$ is connected and we are finished. Thus, we may assume $m \geq 1$.

Let $1 \leq i \leq m$ and consider the component H_i. Since every vertex adjacent to x is in $V(P)$, and H_i is a component of $G - \{x, y\}$ it follows that there exists a vertex $z \in H_i$ adjacent to y. Suppose (for a contradiction) that $|V(H_i)| > 1$. Then since H_i is connected, we may choose a vertex $z' \in V(H_i)$ adjacent to z. But then the graph obtained from $P - x$ by adding the vertices z, z' and the edges yz, zz' is a path longer than P, giving us a contradiction. It follows that H_i consists of a single vertex adjacent only to y.

If $m \geq 2$ and $\{z\} = V(H_1)$ and $\{w\} = V(H_2)$, then z, w have a common neighbor, and $G - \{z, w\}$ is connected, so we are done. Otherwise, $m = 1$ and setting $\{z\} = V(H_1)$ we have that y, z are adjacent vertices, and $G - \{y, z\}$ is connected. This completes the proof.