# Homework 3

**Problem 1.** Let $G$ be a graph with the property that every subgraph of $G$ has a vertex of degree $\leq 1$. Show that $G$ is a forest.

*Solution:* If $G$ contains a cycle $C \subseteq G$, then $C$ has all vertices of degree $\geq 2$, giving us a contradiction. Thus, $G$ is a forest.

**Problem 2.** We proved in class (and in tutorial) that every connected graph $G$ with $|V(G)| \geq 2$ has two vertices $x_1, x_2$ so that $G - x_i$ is connected for $i = 1, 2$. Find an alternative proof of this fact using spanning trees.

*Solution:* Choose a spanning tree $T$ of $G$. We have $|V(T)| \geq 2$ so $T$ has at least two leaf vertices $x_1$ and $x_2$. Now $T - x_i$ is connected for $i = 1, 2$ so $G - x_i$ is also connected for $i = 1, 2$.

**Problem 3.** Let $k \geq 2$ and let $T$ be a tree in which every vertex has degree 1 or degree $k$. Show that $|V(G)| - 2$ is a multiple of $k - 1$.

*Solution:* Suppose that $T$ has $a$ vertices of degree 1 and $b$ vertices of degree $k$. Then

$$(a + b) + |V(G)| - 2 = 2|V(G)| - 2 = 2|E(G)| = \sum_{v \in V(G)} deg(v) = a + bk.$$

Subtracting $a + b$ from both sides shows that $|V(G)| - 2 = b(k - 1)$ which completes the proof.

**Problem 4.** Let $T$ be a tree in which every vertex has degree 1 or degree 3. Prove that there is a vertex adjacent to two leaves.

*Solution:* This is false when $|V(T)| = 2$, but true when $|V(G)| \geq 3$, so we prove it under this assumption. Let $P$ be a longest path in $T$ given by $v_0, e_1, v_1, \ldots, e_n, v_n$. Since $|V(G)| \geq 3$, it follows that $n \geq 2$. Now, consider the vertex $v_1$. Since $v_1$ is adjacent to both $v_0$ and $v_2$, it must have exactly one additional adjacent vertex, call it $u$. Note that $u \notin \{v_0, \ldots, v_n\}$ since $T$ has no cycle. If $u$ is not a leaf vertex, and, say $w$ is adjacent to $u$, then $w \notin \{v_0, \ldots, v_n\}$ (since $T$ has no cycle), but then there is a longer path than $P$ given by $w, wu, u, uv_1, v_1, e_1, \ldots, v_n$. This contradiction shows that $u$ is a leaf. Since $v_0$ must also be a leaf (again by the maximality of our path), it follows that $v_1$ is adjacent to two leaves, as required.

**Problem 5.** Let $n \geq 3$ and let $G$ be an $n$ vertex graph with the property that $G - v$ is a tree for every $v \in V(G)$. What is the graph $G$?

*Solution:* $G$ must be a cycle. First we prove that $G$ is connected. If $G$ has $> 2$ components, then $G - v$ is disconnected for every $v \in V(G)$, a contradiction. If $G$ has exactly two components, then by choosing $v$ to be a vertex in a component with $\geq 2$ vertices we again get the contradiction that $G - v$ is disconnected. Thus $G$ must be connected.

If $G$ has no cycle, then it is a tree, with at least three vertices. If we choose $v$ to be a vertex of degree $\geq 2$, then $G - v$ will be disconnected. Thus, $G$ must have a cycle $C$.

If there is a vertex $v \in V(G) \setminus V(C)$, then $G - v$ has a cycle - namely $C$, which is a contradiction. If there is an edge $e \in E(G) \setminus E(C)$, then removing any vertex of $C$ which is not an endpoint of $e$ will still leave a cycle. It follows that $G = C$, so $G$ is a cycle, as claimed.

**Problem 6.** Let $T$ be a tree with $k$ leaves and set $t = \lceil \frac{k}{2} \rceil$. Prove that there exist paths $P_1, P_2, \ldots, P_t$ which satisfy both properties below.

(i) $\cup_{i=1}^{t} P_i = T$

(ii) $V(P_i) \cap V(P_j) \neq \emptyset$ for every $1 \leq i \leq j \leq t$.

(Hint: first prove that there exist paths satisfying (i))

*Solution:* Let $L$ be the set of leaf vertices. Since every tree is connected, every two points in $L$ are contained in a path. Thus, we may choose a collection of paths $P_1, \ldots, P_t$ so that $L \subseteq \cup_{i=1}^{t} V(P_i)$. Subject to this constraint, choose $P_1, \ldots, P_t$ so that $\sum_{i=1}^{t} |E(P_i)|$ is maximum. We shall show that $P_1, \ldots, P_t$ satisfy (i) and (ii) as well.

Suppose (for a contradiction) that $V(P_i) \cap V(P_j) = \emptyset$. Then we may choose a path $Q \subseteq T$ with ends $u, v$ so that $V(Q) \cap V(P_i) = \{u\}$ and $V(Q) \cap V(P_j) = \{v\}$. But then the subgraph $H = P_i \cup P_j \cup Q$ contains two paths $P_i'$ and $P_j'$ so that $P_i' \cup P_j' = H$ and $P_i' \cap P_j' = Q$. This pair of paths contains the same leaves as $P_i$ and $P_j$ but have more total edges, contradicting our choice. It follows that $P_1, \ldots, P_t$ satisfy (ii).

Suppose (for a contradiction) that $uv \notin \cup_{i=1}^{t} E(P_i)$. Since these paths contain all of the leaves, $u$ and $v$ must not be leaves. But then $T - uv$ is a forest with two components, say $H_1, H_2$ each containing at least one point in $L$. However, then there must exist paths, say $P_i \subseteq H_1$ and $P_2 \subseteq H_2$ for which $V(P_i) \cap V(P_j) = \emptyset$ contradicting (ii). It follows that $P_1, \ldots, P_t$ satisfy (i) as well.

**Problem 7.** Let $d_1, d_2, \ldots, d_n$ be a sequence of positive integers with $\sum_{i=1}^{n} d_i = 2n - 2$. Show that there exists a tree with vertex set $\{x_1, \ldots, x_n\}$ so that $deg(x_i) = d_i$ for every $1 \leq i \leq n$.

*Solution:* We proceed by induction on $n$. As a base, observe that when $n = 2$ there is a unique sequence of positive integers $d_1, d_2$ with $d_1 + d_2 = 2$, namely $d_1 = d_2 = 1$, and $K_2$ is a tree with two vertices of degree one. For the inductive step, we may assume $n \geq 3$. Since $\sum_{i=1}^{n} d_i = 2n - 2 < 2n$, we may assume (without loss) that $d_n = 1$. Similarly, since $\sum_{i=1}^{n} d_i = 2n-2 > n$ we may assume (without loss) that $d_1 > 1$. Now, by applying induction to the sequence $d_1 - 1, d_2, \ldots, d_{n-1}$ we may choose a tree $T$ with vertex set $\{x_1, \ldots, x_{n-1}\}$ so that $deg(x_i) = d_i$ if $2 \leq i \leq n-1$ and $deg(x_1) = d_1 - 1$. Now, add a new vertex $x_n$ to this tree and a new edge $x_1 x_n$. This forms a tree with the required degree properties.