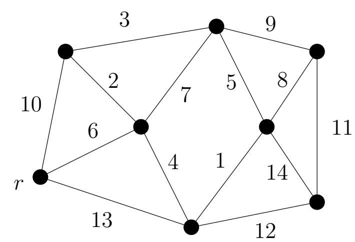
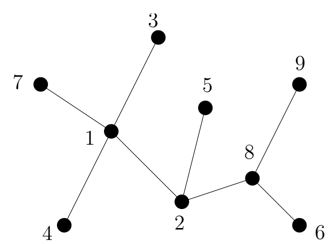
Homework 4

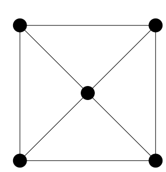
Problem 1. In the weighted graph from the figure below, find the sequence of edge weights selected when both Kruskal's algorithm is run, and when Dijkstra's algorithm is run.



Problem 2. In the tree below with vertex set $\{1, 2, \ldots, 9\}$ apply Prüfer's algorithm to encode this tree, and then apply it to decode this sequence.



Problem 3. Assign the weights $1, 1, 2, 2, 3, 3, 4, 4$ to the edges of the graph below in two ways: one way so that the minimum weight spanning tree is unique, and another way so that the minimum weight spanning tree is not unique.



Problem 4. Let $T = (V, E)$ be a tree with $|V(T)|$ even, and define

$$S = \{e \in E \mid T \setminus e \text{ has two components with an odd number of vertices}\}.$$

Show that every vertex in the graph (V, S) has odd degree.

Problem 5. Let G be a connected graph with weight function $w : E(G) \to \mathbb{R}_+$ and assume that w is one-to-one. If $C \subseteq G$ is a cycle and $e \in E(C)$ is the heaviest edge in C, prove that no minimum weight spanning tree contains the edge e. Use this to prove that the following algorithm produces a minimum weight spanning tree: Iteratively delete the highest weight non-cut-edge until the resulting graph is acyclic.

Problem 6. Let G be a connected graph on n vertices, and let $\mathcal{T}$ be the set of all spanning trees of G. Construct a new graph G' with vertex set $\mathcal{T}$ where $T_1, T_2 \in \mathcal{T}$ are adjacent (as vertices of G') if $|E(T_1) \cap E(T_2)| = n - 2$. Prove that G' is connected.

Problem 7. Let T be a tree and let $T_1, \ldots, T_k \subseteq T$ be trees with the property that $V(T_i) \cap V(T_j) \neq \emptyset$ for every $1 \leq i, j \leq k$. Prove that $\cap_{i=1}^{k} V(T_i)$ is nonempty.