Homework 5

Problem 1. If G is a graph with a maximum matching of size $2k$, what is the smallest possible size of a maximal matching in G?

Problem 2. Prove or disprove: Every tree has at most one perfect matching (a perfect matching is a matching covering every vertex).

Problem 3. Let G be a simple $2n$ vertex graph and assume that every vertex has degree $\geq n+1$. Show that G has a perfect matching.

Problem 4. Let G be a bipartite graph with bipartition (A, B), let $S \subseteq A$ and let $T \subseteq B$. Assume there exist matchings M and M' so that M covers S and M' covers T, and then prove that there exists a matching M^* which covers $S \cup T$.

Problem 5. Let X be a finite set and let $A_1, A_2, \ldots, A_m$ be subsets of X. Prove that one of the following is true

1. There exists a set $I \subseteq \{1, 2, \ldots, m\}$ so that $| \cup_{i \in I} A_i| < |I|$.
2. There exist distinct elements $a_1, a_2, \ldots, a_m \in X$ so that $a_i \in A_i$ for every $1 \leq i \leq m$.

Hint: turn this into a graph theory problem.

Problem 6. Prove that if man m is paired with woman w in some stable marriage, then w does not reject m in the Gale-Shapley Algorithm. Hint: consider the first occurrence of such a rejection.

Problem 7. *Generalizing Tic-Tac-Toe* A positional game consists of a set X of positions and a family $W_1, W_2, \ldots, W_m \subseteq X$ of winning sets (Tic-Tac-Toe has 9 positions corresponding to the 9 boxes, and 8 winning sets corresponding to the three rows, three columns, and two diagonals). Two players alternately choose positions; a player wins when they collect a winning set.

Suppose that each winning set has size at least a and each position appears in at most b winning sets (in Tic-Tac-Toe $a = 3$ and $b = 4$). Prove that Player 2 can force a draw if $a \geq 2b$. Hint: Form a bipartite graph G with bipartition (X, Y) where $Y = \{W_1, W_2, \ldots, W_m\} \cup \{W'_1, W'_2, \ldots, W'_m\}$ with edges xW_j and xW'_j whenever $x \in W_j$. How can Player 2 use a matching in G?