# Homework 5 Solutions

**Problem 1.** If $G$ is a graph with a maximum matching of size $2k$, what is the smallest possible size of a maximal matching in $G$?

*Solution:* The answer is $k$. To construct such a graph, take a graph with $k$ components, each of which is a three edge path. The unique maximum matching uses two edges from each component, but there is a maximal matching using just one from each component. To prove that this is best possible, we need to prove that every graph $G$ with a matching $M^*$ of size $2k$ has the property that every maximal matching $M$ has size at least $k$. To see this, note that in order to be maximal, the matching $M$ must cover at least one endpoint from each edge of $M^*$ (otherwise we could just add this edge to $M$, thus contradicting maximality). It follows that $M$ must cover a set of $2k$ vertices, so it must have size at least $k$.

**Problem 2.** Prove or disprove: Every tree has at most one perfect matching (a perfect matching is a matching covering every vertex).

*Solution:* This is true. Let $M, M'$ be perfect matchings in the tree $T = (V, E)$ and consider the graph on $V$ with edge set $M \cup M'$. Since $M$ and $M'$ both cover all the vertices, every component of this new graph is either a single edge (common to both $M$ and $M'$) or a cycle. Since $T$ is a tree, there can be no cycle, so we conclude that $M = M'$.

**Problem 3.** Let $G$ be a simple $2n$ vertex graph and assume that every vertex has degree $\geq n + 1$. Show that $G$ has a perfect matching.

*Solution:* It follows from Theorem 1.15 that $G$ has a Hamiltonian cycle. Taking every second edge of this cycle yields a perfect matching.

**Problem 4.** Let $G$ be a bipartite graph with bipartition $(A, B)$, let $S \subseteq A$ and let $T \subseteq B$. Assume there exist matchings $M$ and $M'$ so that $M$ covers $S$ and $M'$ covers $T$, and then prove that there exists a matching $M^*$ which covers $S \cup T$.

*Solution:* Consider the graph $H = (V(G), M \cup M')$. Each component of $H$ is either an isolated vertex, an edge which is contained in $M \cap M'$, a cycle with edges alternately in $M$ and $M'$, or a path where edges are alternately from $M$ and $M'$. Let $H_1, \ldots, H_\ell$ be the components of $H$ and choose a matching $M_i$ from each $H_i$ as follows. If $H_i$ is an isolated vertex, then it is not in $S \cup T$ and we let $M_i = \emptyset$. If $H_i$ is either an edge in $M \cap M'$ or a cycle, or a path of odd length, then $H_i$ has a matching $M_i$ which covers $V(H_i)$, so in particular it covers $(S \cup T) \cap V(H_i)$. Finally, we consider the case that $H_i$ is a path of even length. Here the edges must alternate between $M$ and $M'$, so one end of the path is incident with an edge in $M$ and the other in $M'$. However, there must be an odd number of vertices in this path, so either both ends are in $A$ or both ends are in $B$. In the former case we let $M_i = M \cap E(H_i)$ and in the latter we set $M_i = M' \cap E(H_i)$. In either case we again have that all vertices in $S \cup T$ which are contained in $H_i$ are covered by $M_i$. So, now $\cup_{i=1}^{\ell} M_i$ is a matching in $G$ covering $S \cup T$ as desired.

**Problem 5.** Let $X$ be a finite set and let $A_1, A_2, \ldots, A_m$ be subsets of $X$. Prove that one of the following is true

1. There exists a set $I \subseteq \{1, 2, \ldots, m\}$ so that $| \cup_{i \in I} A_i| < |I|$.

2. There exist distinct elements $a_1, a_2, \ldots, a_m \in X$ so that $a_i \in A_i$ for every $1 \leq i \leq m$.

Hint: turn this into a graph theory problem.

*Solution:* Define a simple bipartite graph $G$ with vertex set $\{1, 2, \ldots, m\} \cup X$ and bipartition $(\{1, 2, \ldots, m\}, X)$ by the rule that $i \in \{1, 2, \ldots, m\}$ and $x \in X$ are adjacent if and only if $x \in A_i$. If there exists a matching $M$ in $G$ which covers $\{1, 2, \ldots, m\}$, then for every $1 \leq i \leq m$ let $a_i \in X$ be the element which is paired with $i$ by $M$. Now, by construction $a_1, a_2, \ldots, a_m$ are distinct and $a_i \in A_i$ for every $1 \leq i \leq m$. If there is no such matching, then by Hall's Marriage Theorem, there must exist a set $I \subseteq \{1, 2, \ldots, m\}$ so that $|N(I)| < |I|$. However, then we have $|I| < |N(I)| = | \cup_{i \in I} A_i|$ so the first outcome holds.

**Problem 6.** Prove that if man $m$ is paired with woman $w$ in some stable marriage, then $w$ does not reject $m$ in the Gale-Shapley Algorithm. Hint: consider the first occurrence of such a rejection.

*Solution:* Let $M$ be a stable marriage, and suppose for a contradiction that during the Gale-Shapley algorithm, some man $m$ is rejected by a woman $w$ for which $m$ and $w$ are paired in $M$. Consider the first step of the algorithm during which such a rejection occurs. Since $m$ is rejected by $w$ on this step, $w$ must receive a proposal from some man $m'$ whom she prefers to $m$ on this step. Since, by assumption $M$ is a stable marriage, it follows that $m'$ must be paired with a woman $w'$ in $M$ with the property that $m'$ prefers $w'$ to $w$. However, since $m'$ is proposing to $w$ at this step of the Gale-Shapley algorithm, he must already have been rejected by $w'$, but this contradicts our assumption that this was the first step of the algorithm on which a rejection of the given type occurs.

**Problem 7.** *Generalizing Tic-Tac-Toe* A positional game consists of a set $X$ of positions and a family $W_1, W_2, \ldots, W_m \subseteq X$ of winning sets (Tic-Tac-Toe has 9 positions corresponding to the 9 boxes, and 8 winning sets corresponding to the three rows, three columns, and two diagonals). Two players alternately choose positions; a player wins when they collect a winning set.

Suppose that each winning set has size at least $a$ and each position appears in at most $b$ winning sets (in Tic-Tac-Toe $a = 3$ and $b = 4$). Prove that Player 2 can force a draw if $a \geq 2b$. Hint: Form a bipartite graph $G$ with bipartition $(X, Y)$ where $Y = \{W_1, W_2, \ldots, W_m\} \cup \{W'_1, W'_2, \ldots, W'_m\}$ with edges $xW_j$ and $xW'_j$ whenever $x \in W_j$. How can Player 2 use a matching in $G$?

*Solution:* Let $Y' \subseteq Y$ and define $S$ to be the set of all edges incident with a vertex in $Y'$. Since every vertex in $Y$ has degree at least $a$ we must have $|S| \geq a|Y'|$. On the other hand, every vertex in $X$ has degree at most $2b$, so we must have $2b|N(Y')| \leq |S|$. Combining these gives us $a|Y'| \leq 2b|N(Y')|$ and together with the assumption $a \geq 2b$ we find $|N(Y')| \geq |Y'|$. So, it now follows from Hall's Theorem that there is a matching $M$ which covers $Y$.

For every $1 \leq i \leq m$ let $Q_i$ be the set consisting of the two vertices in $X$ which are matched to $W_i$ and $W'_i$. Now the sets $Q_1, \ldots, Q_m$ are disjoint two element subsets of $X$ and every $W_i$ contains $Q_i$. Here is a strategy which will guarantee the second player a draw (or better). For each move made by the first player, if the first player chooses a position $x \in Q_i$ for some $1 \leq i \leq m$ then the second player responds by choosing the other position in $Q_i$ if it is available. Otherwise, the second player just plays arbitrarily. It follows from a straightforward induction that after every turn of the second player, there is no set $Q_i$ for which player 1 has chosen one element, and player 2 none. It follows from this that player 1 can never choose both members of a set $Q_i$, and from this that player 1 cannot choose all members of any $W_i$. Thus, player 1 cannot win when player 2 adopts this strategy.