Homework 6 Solutions

1. Exhibit a Marriage System which has more than one stable marriage.

Solution: Let $(\{a,b\},\{x,y\})$ be a bipartition of $K_{2,2}$ and define a system of preferences as follows:

$$\begin{aligned} a &: x > y \\ b &: y > x \\ x &: b > a \\ y &: a > b \end{aligned}$$

Both perfect matchings of this graph are stable marriages: $\{ax, by\}$ is stable since a and b are both adjacent to the vertex they most prefer. On the other hand, $\{ay, bx\}$ is stable since y and x are both adjacent to the vertex they most prefer.

2. Use Theorem 11.2 (König Egerváry) to prove Theorem 9.2 (Hall's Marriage Theorem).

Solution: The "only if" direction is obvious. For the "if" direction, Let G be bipartite with bipartition (A, B). If there is a matching of size $|A|$, then this matching covers A and we are finished. Otherwise, $\beta(G) = \alpha'(G) < |A|$ so we may choose a vertex cover $Y \subseteq V(G)$ with $|Y| < |A|$. Since Y is a vertex cover, $N(A \setminus Y) \subseteq B \cap Y$. But then we have $|A \setminus Y| = |A| - |Y \cap A| > |Y \cap B| \geq |N(A \setminus Y)|$, which completes the proof. □

Recall that a *permutation matrix* is a square matrix with all entries 0 or 1 in which every row and every column has exactly one 1.

3. Let Q be an $n \times n$ nonnegative real matrix, let $t \in \mathbb{R}$, and assume that every row and every column sum to t. Prove that there exist permutation matrices $P_1, \ldots, P_k$ and real numbers $x_1, \ldots, x_k$ so that $Q = \sum_{i=1}^{k} x_i P_i$. (Hint: proceed by induction on the number of nonzero entries in Q, and show that there is a suitable permutation matrix P).

Solution: We proceed by induction on the number of nonzero entries of Q. As a base, observe that if Q is the zero matrix, then Q is an empty sum of permutation matrices. For the inductive step, we may assume that Q has at least one nonzero entry. Let A be the set of rows of Q and B be the set of columns, and define a simple bipartite graph G with bipartition (A, B) by the rule that $a \in A$ is adjacent to $b \in B$ if the (a, b) entry in Q is nonzero. Let

$X \subseteq A$. By construction, every entry in the submatrix $Q^X_{B \setminus N(X)}$ is zero, so by summing up the entries in each row, we see that the sum of the entries in the submatrix $Q^X_{N(X)}$ is exactly equal to $t|X|$. On the other hand, by summing up the entries in each column, we see that the sum of the entries in this submatrix is at most $t|N(X)|$ (here we use the assumption that all entries are nonnegative). It then follows that $|N(X)| \geq |X|$ (here we use $t > 0$). Since this is true for every $X \subseteq A$, it follows from Hall's theorem that G has a perfect matching. This corresponds to a set of n entries from Q with exactly one from each row and column. Let P be the corresponding permutation matrix, and let x be the smallest value of Q appearing in one of these entries. Now the matrix $Q' = Q - xP$ is a nonnegative matrix in which every row and column sum to $t - x$ which has more zero entries than Q. Thus, by induction, there exist real numbers $x_1, \ldots x_k$ and permutation matrices $P_1, \ldots, P_k$ with $Q' = \sum_{i=1}^k x_i P_i$, and we then have $Q = xP + \sum_{i=1}^k x_i P_i$ as desired.

4. For every $k \geq 1$ construct a simple graph in which every vertex has degree $2k$ which has no perfect matching.

Solution: K_{2k+1} is such a graph.

5. For every $k \geq 1$ construct a simple graph in which every vertex has degree $2k + 1$ which has no perfect matching. (Hint: start with $k = 1$ and then generalize).

Solution: For every $k \geq 1$ let H_k be the graph obtained from a copy of $K_{2k,2k+1}$ by adding a k-edge matching to the partite set of size $2k + 1$. Then H_k has $2k$ vertices of degree $2k + 1$ and 1 vertex of degree $2k$. Now, form a graph G_k by taking $2k+1$ copies of H_k, then adding one new vertex, say v, and then adding an edge between v and each vertex of degree $2k$ in each copy of H_k. Then G_k is $(2k + 1)$-regular, but cannot have a perfect matching, since $odd(G - \{v\}) = 2k + 1 > 1 = |\{v\}|$.

6. Prove that a tree T has a perfect matching if and only if $odd(T - v) = 1$ for every $v \in V(T)$.

Solution: If T has a perfect matching, then $|V(T)|$ is even, so $T - v$ must have an odd number of components for every v. On the other hand, since T has a perfect matching, $odd(T - v) \leq 1$. Thus $odd(T - v) = 1$ for every $v \in V(T)$.

Next, assume that $odd(T - v) = 1$ for every $v \in V(T)$ and set $M = \{e \in E(T) : odd(T - e) = 2\}$. Let $v \in V(T)$ and let u be the (unique) vertex in the odd component of

$T - v$ which is adjacent to v. It follows that $uv \in M$, and that this is the only edge incident with v in M. Since this holds for every v, we find that M is a perfect matching.

7. Let G be a simple graph in which every vertex has degree 3. Prove that G has a perfect matching if and only if there is a decomposition of G into 3-edge paths.

Solution: First suppose that G has a decomposition into three edge paths $P_1, \ldots, P_k$. Let M be the collection of edges which appear as the middle edge in one of $P_1, \ldots, P_k$. Now $2k = 6|E(G)| = 3\sum_{v \in V(G)} deg(v) = |V(G)|$, so $|M| = \frac{1}{2}|V(G)|$. Since $P_1, \ldots, P_k$ form a decomposition of G, it follows that no two edges of M share an endpoint. Since $|M| = \frac{1}{2}|V(G)|$, it then follows that M is a perfect matching.

Next suppose that M is a perfect matching of G, and let $H = G - M$. Then H is 2-regular, so every component of H is a cycle. Assign a "clockwise" orientation to each cycle of H. Now for each edge $uv \in M$, complete M to a 3-edge path by adding the two edges which follow u and v in the clockwise ordering of the cycles in H. This gives a decomposition of G into 3-edge paths as required.