Homework 7 Solutions

Problem 1. Find a graph on 100 vertices with ≥ 98 cut vertices.

Solution: P_{100}.

Problem 2. Find a formula for the number of spanning trees of a graph in terms of the number of spanning trees of each block.

Solution: Let G be a graph and let (G_1, G_2) be a 1-separation of G. If T_i is a spanning tree of G_i for $i = 1, 2$ then $T = T_1 \cup T_2$ is a connected graph with no cycle with vertex set $V(G)$, so T is a spanning tree of G. Conversely, if T is a spanning tree of G, then for $i = 1, 2$ define $T_i = (V(G_i) \cap V(T), E(G_i) \cap V(T)$. Now each T_i is a subgraph of G_i which has no cycles and contains all vertices of G_i. Furthermore, if $x, y \in V(G_i)$ then there is a path in T from x to y, and this path must be contained in T_i (it cannot "cross" the 1-separation), so T_i is a spanning tree of G_i. This establishes a bijection between the spanning trees of G and pairs of spanning trees of G_1 and G_2.

Now, for every graph G we define the function $ST(G)$ to be the number of spanning trees of G. It follows from the above that $ST(G) = ST(G_1) \cdot ST(G_2)$ whenever (G_1, G_2) is a 1-separation of G. We claim that for every graph G with blocks $B_1, \ldots, B_m$ that the formula $ST(G) = \prod_{i=1}^{m} ST(B_i)$ holds. We prove this by induction on m. As a base case, observe that this holds automatically when G consists of a single block. For the inductive step, we may assume that G has a cut vertex x. In this case, it follows from the Block-Cutpoint decomposition that we may partition the blocks of G into two sets, without loss $\{B_1, \ldots, B_k\}$ and $\{B_{k+1}, \ldots, B_m\}$ so that G has a 1-separation (G_1, G_2) with $V(G_1) \cap V(G_2) = \{x\}$ so that $B_1, \ldots B_k$ are the blocks of G_1 and $B_{k+1} \ldots, B_m$ are the blocks of G_2. Now using our earlier observation together with induction gives us $ST(G) = ST(G_1) \cdot ST(G_2) = \prod_{i=1}^{k} ST(B_i) \cdot \prod_{i=k+1}^{m} ST(B_i) = \prod_{i=1}^{m} ST(B_i)$ as desired.

Problem 3. If C_1 and C_2 are cycles of maximum length in a 2-connected graph, show that $|V(C_1) \cap V(C_2)| \geq 2$.

First suppose (for a contradiction that $V(C_1) \cap V(C_2) = \{x\}$. Now $G - x$ is connected, so this graph contains a path with initial vertex in $V(C_1) - x$ and terminal vertex in $V(C_2) - x$. If we choose a minimal such path P, then all internal vertices of P will be disjoint from $V(C_1) \cup V(C_2)$. However, then we find that the graph $C_1 \cup C_2 \cup P$ contains a cycle longer than C_1 or C_2, which is a contradiction (if $\{y_i\} = V(P) \cap V(C_i)$ then form a cycle C by taking the path P together with a longest path in C_1 from y_1 to x and a longest path in C_2 from y_2 to x).

Next we suppose (for a contradiction) that $V(C_1) \cap V(C_2) = \emptyset$. Now choose an edge $e \in E(C_1)$ and an edge $F \in E(C_2)$ and choose a cycle D which contains e and f. Now $D - \{e, f\}$ contains two paths P and Q from $V(C_1)$ to $V(C_2)$ so there exist two vertex disjoint paths P', Q' from $V(C_1)$ to $V(C_2)$ which have no internal vertices in $V(C_1) \cup V(C_2)$. Again, we now have that $C_1 \cup C_2 \cup P' \cup Q'$ contains a longer cycle than C_1 or C_2, which is a contradiction (if $V(C_i) \cap V(P') = \{x_i\}$ and $V(C_i) \cap V(Q') = \{y_i\}$ then we form a longer cycle by taking $P' \cup Q'$ together with a longest path in C_1 from x_1 to y_1 and a longest path in C_2 from x_2 to y_2).

Problem 4. Let G be a simple graph, and assume that every vertex has degree $\geq |V(G)| - 2$. Prove that $G - X$ is connected whenever $|X| < |V(G)| - 2$.

Solution: Suppose (for a contradiction) that $G - X$ is disconnected for some $X \subseteq V(G)$ with $|X| \leq |V(G)| - 3$. Now $G - X$ has at least three vertices, and at least two components, so we may choose a component H of $G - X$ so that $|V(H)| + |X| \leq |V(G)| - 2$. However, now there exist two distinct vertices $x, y \in V(G - X)$ which are not in the component H. But then for $w \in V(H)$ we have that w is not adjacent to x or y, so $deg(w) \leq |V(G)| - 3$ which contradicts our assumption.

Problem 5. Prove that every vertex of G has even degree if and only if every block of G is Eulerian.

Solution: If every block of G is Eulerian, then since G is the edge-disjoint union of its blocks, every vertex in G must have even degree. For the other direction, we must prove that whenever G has all degrees even and has blocks $B_1, \ldots, B_m$ then every B_i is Eulerian. We shall prove this by induction on m. When $m = 1$ the result follows from the definition. For the inductive step, we may assume that G is connected and that G has at least two blocks. Then we may choose a cut vertex x and a 1-separation (G_1, G_2) of G with $V(G_1) \cap V(G_2) = \{x\}$. By assumption all vertices in G_i except possibly x have even degree. However, every graph has an even number of vertices of odd degree, so we find that all vertices in G_i have even degree. Now, by induction, every block in G_1 or G_2 is Eulerian, and thus the same holds for G.

Definition: For a set of vertices A we let $\delta(A) = \{uv \in E(G) \mid\ u \in A \text{ and } v \notin A\ \}$ and we call any set of the form $\delta(A)$ an *edge cut*.

Problem 6. Prove that the symmetric difference of two edge cuts is an edge cut.

Solution: For edge cuts $R = \delta(A)$ and $S = \delta(B)$, the symmetric difference of R and S is given by the edge cut $\delta((A \cup B) \setminus (A \cap B))$. To see this, note that an edge e satisfies $e \in \delta((A \cup B) \setminus (A \cap B))$ if and only if e is in exactly one of $\delta(A)$ and $\delta(B)$.

Problem 7. Let F be a nonempty set of edges in G. Prove that F is an edge cut if and only if F contains an even number of edges from every cycle in G. For example, when $G = C_n$ every even subset of edges is an edge cut, but no odd set of edges is an edge cut. Hint: For sufficiency, the task is to show that the components of $G - F$ can be grouped into two nonempty collections so that every edge in F has an endpoint in each collection.

Solution: Let $H_1, \ldots, H_m$ be the components of $G - F$ and construct a new graph $\tilde{G}$ with vertex set $\{H_1, \ldots, H_m\}$ and edge set F where the edge $uv \in F$ has ends H_i and H_j if $u \in V(H_i)$ and $v \in V(H_j)$. Suppose (for a contradiction) that $\tilde{G}$ contains an odd cycle (so $\tilde{(G)}$ is not bipartite). Now by Theorem 1.9 we may choose an induced odd cycle C, and we may assume without loss that C has vertices $H_1, \ldots, H_{2k-1}$ and edge set $S \subseteq F$ with $|S| = 2k-1$. For every $1 \leq i \leq 2k-1$ there will be exactly two edges in S which are incident with vertices in $V(H_i)$ and we let x_i, y_i denote these two vertices. Since H_i is connected, we may choose a path $P_i \subseteq H_i$ from x_i to y_i for each $1 \leq i \leq k$. Now the graph consisting of $\cup_{i=1}^{2k-1} P_i + S$ is a cycle of the original graph with an odd number of edges in S, which is a contradiction. Therefore the graph $\tilde{G}$ is bipartite. So we may choose a partition of the components $H_1, \ldots, H_m$ into two sets, say $\{\{H_1, \ldots, H_\ell\}, \{H_{\ell+1} \ldots, H_m\}\}$ so that every edge in F has one end in the set $A = \cup_{i=1}^{\ell} V(H_i)$ and the other end in the set $B = \cup_{i=\ell+1}^{m} V(H_i)$. It now follows that F is precisely the edge cut given by $\delta(A) = \delta(B)$.