Homework 8 Solutions

Problem 1. Find a 100-connected bipartite graph G for which $|V(G)|$ is minimum.

Solution: The graph $K_{100,100}$ is the smallest 100-connected bipartite graph. Every 100-connected graph has minimum degree ≥ 100, so every 100-connected bipartite graph with bipartition (A, B) must have $|A| \geq 100$ and $|B| \geq 100$. It follows that every 100-connected bipartite graph G satisfies $|V(G)| \geq 200 = |V(K_{100,100})|$.

Problem 2. Prove or Disprove: If G is a 2-connected graph and $P \subseteq G$ is a path from u to v, then $G - E(P)$ contains a path from u to v.

Solution: This is false. For a counterexample, consider the graph G obtained from K_4 by deleting an edge, and let u, v be the two vertices of degree 2 in G. It is immediate that G is 2-connected. Furthermore, G contains a path P of length 3 from u to v, and $G - E(P)$ has no path from u to v.

Problem 3. Prove or Disprove: If G is a 2-connected graph and $x, y, z \in V(G)$, then there exists a path from x to z which contains y.

Solution: This is true. To prove it, construct a new graph G' from G by adding a new edge $e = xz$, and note that G' is 2-connected. Now choose an edge f incident with y and apply Theorem 4.4 to choose a cycle $C \subseteq G'$ with $e, f \in G'$. Then $P = G \setminus e$ is a path in G from x to z which contains y.

Problem 4. Let G be a connected graph with $|V(G)| \geq 2$, and assume that G has no cycle of even length. Prove that every block of G is either an edge or an odd cycle.

Solution: It follows from the observation that every block of G is either a 2-connected graph or a graph on at most two vertices that to solve the problem it suffices to prove:

- If H is 2-connected with no cycle of even length, then H is a cycle of odd length.

To prove this, we choose a cycle $C \subseteq H$. If $C = H$ the result is immediate, so we may assume there exists an edge $e \in E(H) \setminus E(C)$. Choose an edge $f \in E(C)$ and apply Theorem 4.4 to choose a cycle $D \subseteq H$ with $e, f \subseteq E(D)$. Now let $P_0 \subseteq D$ be the maximal path containing the edge e with the property that all interior vertices of P_0 are not contained in C. Let u, v be the ends of P_0 and let P_1, P_2 be the two paths in C from u to v. Now, either two of the paths P_0, P_1, P_2 have even length, or two of them have odd length. Combining this pair yields a cycle of H with even length.

Problem 5. Let v be a vertex of a 2-connected graph G. Prove that v has a neighbour u so that $G - \{u, v\}$ is connected.

Solution: Let $N = N(v)$ and define the graph $G' = G - v$. If G' is 2-connected, then for every $u \in N$ we have that $G - \{u, v\} = G' - u$ is connected, which solves the problem. Otherwise, G' is connected, but not 2-connected, and we consider the block-cutpoint graph of G'. Let H be a block of G' which is a leaf vertex of the block cutpoint tree, and let $w \in V(H)$ be the unique cut vertex of G' which is contained in $V(H)$. Now w is not a cut-vertex of the original graph G, since G is 2-connected. It follows from this that there exists $u \in V(H) \cap N$. Now, by assumption the graph $H - u$ is connected, and it follows from this that $G - \{u, v\} = G' - u$ is connected.

Problem 6. Let G be a connected graph with no cut-edge. Define a binary relation $\sim$ on $E(G)$ by the rule that $e, f \in E(G)$ satisfy $e \sim f$ if either $e = f$ or $G - \{e, f\}$ is disconnected.

1. Show that $e \sim f$ if and only if e and f belong to the same cycles (i.e. for every cycle C, either C contains both e and f or it contains neither).

2. Show that $\sim$ is an equivalence relation.

3. For each equivalence class F, show that there is a cycle containing all of F.

Solution: First observe the following property which holds for all $e, f \in E(G)$.

$$(\star) \quad e \text{ is a cut-edge of } G - f \;\Leftrightarrow\; e \sim f \;\Leftrightarrow\; f \text{ is a cut-edge of } G - e$$

For the first part, note that if e, f are contained in the same cycles, then f must be a cut-edge of $G - e$ so $e \sim f$. On the other hand, if $e \sim f$ then $G - e$ has no cycle containing f and $G - f$ has no cycle containing e, so e, f are contained in the same cycles.

For the second part, note that our definitions immediately imply that $\sim$ is both reflexive and symmetric. To see that it is transitive, let $e \sim e'$ and $e' \sim e''$. Now choose a cycle C containing e. Since $e \sim e'$ we must have $e' \in E(C)$, but then $e' \sim e''$ implies $e'' \in E(C)$. It follows that $e \sim e''$, so $\sim$ is an equivalence relation.

Let F be the set of edges equivalent to an edge f and choose a cycle $C \subseteq G$ with $f \in E(C)$. Since every edge in F is equivalent to f we have $F \subseteq E(C)$, as desired.

Problem 7. Let G be a 3-regular 3-connected graph and let $u, v \in V(G)$. Prove that G contains a path P from u to v with the property that $G - V(P)$ is connected. (Hint: choose a path P from u to v so that in the graph $G - V(P)$ the largest component is as large as possible, and subject to this the second largest component is as large as possible, and so on.)

Solution: Following the hint, choose a path P from u to v with the property that the largest component H_1 of $G - V(P)$ is as large as possible, and subject to this, the second largest component G_2 of $G - V(P)$ is as large as possible, and so on, letting H_m denote the smallest component.

If P is not an induced path, i.e. there is an edge $e \in E(G) \setminus E(P)$ with both ends in $V(P)$ then there is a shorter path P' from u to v contained in $P + e$ and this path contradicts our choice of P. So, we may assume that no such edge exists.

If there is just one component, then we are finished, so we may assume this is not the case. Let H_m be the smallest component and let Q be the minimal path contained in P which contains all vertices in $N(V(H_m))$. Let x, y be the ends of Q and note that since $G - \{x, y\}$ is connected, there must be another component H_i with the property that $N(V(H_i))$ contains a vertex in the interior of the path Q. Next choose a path R from x to y so that all interior vertices of R are contained in $V(H_m)$. By rerouting the original path P along R, we obtain a new path from u to v which has increased the size of the component H_i. This path contradicts the choice of P, thus completing the proof.