# Homework 9

**Problem 1.** Find a strongly connected directed graph with 100 vertices and the fewest possible edges.

**Problem 2.** Construct a tournament on 10 vertices with no directed cycle of length 3.

**Problem 3.** Let $D$ be a directed graph whose underlying graph is connected. Let $u_1, u_2 \in V(D)$ and assume that $deg^+(u_1) > deg^-(u_1)$ and that $deg^+(v) = deg^-(v)$ for every vertex $v \in V(D) \setminus \{u_1, u_2\}$. Prove that $D$ contains a directed walk from $u_1$ to $u_2$.

**Problem 4.** Show that every tree can be oriented so that in the resulting digraph every vertex $v$ satisfies $|deg^+(v) - deg^-(v)| \leq 1$.

**Problem 5.** Show that every graph can be oriented so that in the resulting digraph every vertex $v$ satisfies $|deg^+(v) - deg^-(v)| \leq 1$.

**Problem 6.** Let $k$ be an integer and let $D$ be a directed graph with the property that $deg^+(v) = k = deg^-(v)$ for every $v \in V(D)$. Prove that there exist vertex disjoint directed cycles $C_1, \ldots, C_t$ so that $\cup_{i=1}^{t} V(C_i) = V(D)$. (Hint: construct a bipartite graph $H$ from $D$ so that each vertex in $D$ splits into two vertices in $H$.)

**Problem 7.** Let $D$ be a strongly connected orientation of the graph $G$. Prove that if $G$ has a cycle of odd length, then $D$ has a directed cycle of odd length. (Hint: consider each pair $\{v_i, v_{i+1}\}$ in an odd cycle of $G$ with vertices $v_1, \ldots, v_k$).