Homework 9 Solutions

Problem 1. Find a strongly connected directed graph with 100 vertices and the fewest possible edges.

Solution: Let D be a directed cycle on 100 vertices. If D' is any strongly connected digraph with $|E(D')| = 100$, then every vertex in D' must have positive outdegree, and this implies $|E(D')| \geq 100 = |E(D)|$. Thus D is a strongly connected digraph on 100 vertices with the fewest number of edges.

Problem 2. Construct a tournament on 10 vertices with no directed cycle of length 3.

Solution: Let D be a digraph with vertex set $\{v_1, v_2, \ldots, v_{10}\}$ and edges (v_i, v_j) whenever $i < j$. Then D is a tournament with no directed cycles.

Problem 3. Let D be a directed graph whose underlying graph is connected. Let $u_1, u_2 \in V(D)$ and assume that $deg^+(u_1) > deg^-(u_1)$ and that $deg^+(v) = deg^-(v)$ for every vertex $v \in V(D) \setminus \{u_1, u_2\}$. Prove that D contains a directed walk from u_1 to u_2.

Solution: Let $k = deg^+(u_1) - deg^-(u_1)$ and construct a new digraph D' from D by adding k new edges of the form (u_2, u_1). In the digraph D' we have $deg^+_{D'}(v) = deg^-_{D'}(v)$ for every $v \in V(D') \setminus \{u_2\}$. Furthermore, $\sum_{v \in V(D')} deg^+_{D'}(v) = |E(D')| = \sum_{v \in V(D')} deg^-_{D'}(v)$ which implies that $deg^+_{D'}(v) = deg^-_{D'}(v)$ for every $v \in V(D')$. It now follows from Theorem 5.11 that D' has an Eulerian walk W. By shifting the starting point, we may assume that W starts at the vertex u_1. Now let W^* be the walk obtained from W by starting at the vertex u_1 and following the walk W until the first occurrence of an edge in D' but not D. This new walk W^* is a walk in D from u_1 to u_2 as desired.

Problem 4. Show that every tree can be oriented so that in the resulting digraph every vertex v satisfies $|deg^+(v) - deg^-(v)| \leq 1$.

Solution: We prove by induction on the number of vertices that every tree has an orientation satisfying the following property

$$(\star) \quad |deg^+(v) - deg^-(v)| \leq 1$$

As a base case, note that the result is trivial for a 1-vertex tree. For the inductive step, let T be a tree with $|V(T)| \geq 2$ and assume the result holds true for every tree with fewer than

$|V(T)|$ vertices. Now choose a leaf vertex $v \in V(T)$ and let $T' = T - v$. By induction we may choose an orientation of T' satisfying $(\star)$. Let u be the unique neighbour of v in T. Now, we construct an orientation of T by taking the orientation of the edges of T' and then choosing an orientation of uv. Since u satisfies rule $(\star)$ in T' we may choose an orientation of uv so that u also satisfies $(\star)$ in T. Since v automatically satisfies $(\star)$ in T this solves the problem.

Problem 5. Show that every graph can be oriented so that in the resulting digraph every vertex v satisfies $|deg^+(v) - deg^-(v)| \leq 1$.

Solution: We prove by induction on $|E(G)|$ that every graph G has an orientation satisfying rule $(\star)$ from the previous problem. As a base case, note the result holds trivially when $|E(G)| = 0$. For the inductive step, let G be a graph with $|E(G)| > 0$ and assume the result holds for every graph with fewer than $|E(G)|$ edges. If G does not have a cycle, then the result holds by applying the previous problem to each component of G. So, we may assume G contains a cycle C. Let $G' = G - E(C)$ and apply induction to choose an orientation of G' satisfying $(\star)$. Now we orient the edges of G by taking this orientation of G' and then orienting the edges in $E(C)$ so that C is a directed cycle. It follows that this orientation satisfies $(\star)$.

Problem 6. Let k be an integer and let D be a directed graph with the property that $deg^+(v) = k = deg^-(v)$ for every $v \in V(D)$. Prove that there exist vertex disjoint directed cycles $C_1, \ldots, C_t$ so that $\cup_{i=1}^t V(C_i) = V(D)$. (Hint: construct a bipartite graph H from D so that each vertex in D splits into two vertices in H.)

Solution: We construct a bipartite graph H with bipartition (A, B) as follows. For each vertex $v \in V(D)$ let $v_1 \in A$ and let $v_2 \in B$. Now for every edge $(u, v) \in E(D)$ we add the edge $u_1 v_2$ to H. It follows from our assumptions that H is a k-regular bipartite graph with bipartition (A, B). Every regular bipartite graph has a perfect matching (Corollary 3.3) so we may choose a perfect matching M of H. Now form a set of edges S in D by the rule that for every $u_1 v_2 \in M$ we add the edge (u, v) to S. It follows immediately that every vertex $w \in V(D')$ satisfies $|\delta^+(w) \cap S| = 1 = |\delta^-(w)|$. Therefore, the digraph $D - (E(D) \setminus S)$ is a disjoint union of directed cycles $C_1, \ldots, C_t$ as desired.

Problem 7. Let D be a strongly connected orientation of the graph G. Prove that if G has a cycle of odd length, then D has a directed cycle of odd length. (Hint: consider each pair $\{v_i, v_{i+1}\}$ in an odd cycle of G with vertices $v_1, \ldots, v_k$).

Solution: We shall first prove that D has a closed directed walk of odd length. Let $v_1, v_2, \ldots, v_m$ be a cyclic ordering of vertices which form a cycle in the underlying graph G. Now, (working modulo m), consider each pair of vertices v_i, v_{i+1}. If the edge between them is oriented from v_i to v_{i+1}, let W_i be the directed path consisting of these two vertices and this one edge. If this edge is oriented from v_{i+1} to v_i, then since D is strongly connected, we may choose W_i to be a directed path from v_{i+1} to v_i. If W_i has even length, then W_i together with the edge (v_{i+1}, v_i) is an odd directed cycle and we are done. Thus, we may assume that W_i has odd length. So, in other words, we have constructed for each i a directed walk from v_i to v_{i+1} of odd length. Concatenating these walks (in the obvious manner) yields a closed directed walk of odd length. Now, by minimality, every closed directed walk of odd length contains a directed cycle of odd length. This completes the answer.