3 Matchings

Hall's Theorem

Matching: A *matching* in G is a subset $M \subseteq E(G)$ so that no edge in M is a loop, and no two edges in M are incident with a common vertex. A matching M is *maximal* if there is no matching M' with $M \subset M'$ and *maximum* if there is no matching M'' with $|M| < |M''|$.

Alternating & Augmenting Paths: If M is a matching in G, a path $P \subseteq G$ is M-*alternating* if the edges of P belong alternately to M and to $E(G) \setminus M$ (in other words, for every $v \in V(P)$ with degree 2 in P, some edge of P incident with v is in M). The path P is M-*augmenting* if it is M-alternating, has distinct ends, say u, v, and no edge of M is incident with u or v in G (not just in P).

Theorem 3.1 (Berge) *A matching M in G is maximum if and only if there is no M-augmenting path.*

Proof: For the "only if" direction we prove the contrapositive. Assuming G contains an M-augmenting path P, the set $(M \setminus E(P)) \cup (E(P) \setminus M)$ is a matching with larger cardinality than M, so M is not maximum.

For the "if" direction, we also prove the contrapositive, so we shall assume that M is not maximum, and show there is an augmenting path. Since M is not maximum, there exists a matching M' with $|M'| > |M|$. Consider the subgraph $H \subseteq G$ with $V(H) = V(G)$ and $E(H) = M \cup M'$. Every component of this graph is either a cycle of even length with edges alternately in M and M', a path with edges alternately in M and M', or a path consisting of one edge e with $e \in M \cap M'$. Since $|M'| > |M|$, there is a component of H which is a path with more edges in M' than M. Then P is an M-augmenting path. $\square$

Neighbors: If $X \subseteq V(G)$, the *neighbors* of X, is the set

$$N(X) = \{v \in V(G) \setminus X : v \text{ is adjacent to some point in } X\}.$$

For $x \in X$, we define $N(x) = N(\{x\})$.

Cover: We say that a set of edges $S \subseteq E(G)$ *covers* a set of vertices X if every $x \in X$ is incident with some edge in S. Similarly, a set of vertices $X \subseteq V(G)$ *covers* a set of edges $S \subseteq E(G)$ if every edge in S is incident with some point in X.

Theorem 3.2 (Hall's Marriage Theorem) *Let G be a bipartite graph with bipartition (A, B). Then, there is a matching $M \subseteq G$ which covers A if and only if $|N(X)| \geq |X|$ for every $X \subseteq A$.*

Proof: The "only if" condition is obvious, if there exists $X \subseteq A$ with $|N(X)| < |X|$, then no matching can cover A.

For the "if" direction, let M be a maximum matching, and suppose that M does not cover A. Choose a vertex $u \in A$ not covered by M, and define the sets X, Y as follows:

$$X = \{x \in A : \text{there is an } M\text{-alternating path from } u \text{ to } x\}$$
$$Y = \{y \in B : \text{there is an } M\text{-alternating path from } u \text{ to } y\}$$

Let $y \in Y$ and choose an M-alternating path P from u to y. Note that by parity, the last edge of P is not in M. Now, P cannot be M-augmenting, so there must exist an edge in M incident with y, call it yx. We claim that $x \in X$. This is immediate if $x \in V(P)$. Otherwise, we may extend P by the edge xy to get an M-alternating path ending at x. Thus, every point in Y is joined by an edge in M to a point in $X \setminus \{u\}$. It follows that $|Y| < |X|$.

Let $x \in X$ and let $y \in N(x)$. Since $x \in X$, we may choose an M-alternating path P from u to x. Note that the last edge of this path is in M. We claim that $y \in Y$. This is immediate if $y \in V(P)$. Otherwise, $xy \notin M$ (why?), so we may extend P by the edge xy to obtain a new M-alternating path ending at y. Thus, in either case, we find that $y \in Y$. It follows from this that $N(X) \subseteq Y$. But then, $|N(X)| \leq |Y| < |X|$. This completes the proof. □

Regular: A graph G is *k-regular* if every vertex of G has degree k. We say that G is *regular* if it is k-regular for some k.

Perfect Matchings: A matching M is *perfect* if it covers every vertex.

Corollary 3.3 *Every regular bipartite graph has a perfect matching.*

Proof: Let G be a k-regular bipartite graph with bipartition (A, B). Let $X \subseteq A$ and let t be the number of edges with one end in X. Since every vertex in X has degree k, it follows that $k|X| = t$. Similarly, every vertex in $N(X)$ has degree k, so t is less than or equal to $k|N(X)|$. It follows that $|X|$ is at most $|N(X)|$. Thus, by Hall's Theorem, there is a matching covering A, or equivalently, every maximum matching covers A. By a similar argument, we find that every maximum matching covers B, and this completes the proof. □

Covers

Covers: A *vertex cover* of G is a set of vertices $X \subseteq V(G)$ so that every edge is incident with some vertex in X. Similarly, an *edge cover* of G is a set of edges $S \subseteq V(G)$ so that every vertex is incident with some edge in S.

Independent Set: A subset of vertices $X \subseteq V(G)$ is *independent* if there is no loop with endpoint in X and there is no non-loop with both ends in X.

Matching & Cover Parameters: For every graph G, we define the following parameters

$\alpha(G)$	maximum size of an independent set
$\alpha'(G)$	maximum size of a matching
$\beta(G)$	minimum size of a vertex cover
$\beta'(G)$	minimum size of an edge cover

Observation 3.4 *$\alpha(G) + \beta(G) = v(G)$ for every simple graph G.*

Proof: A set $X \subseteq V(G)$ is independent if and only if $V(G) \setminus X$ is a vertex cover. Thus, the complement of an independent set of maximum size is a vertex cover of minimum size. □

Theorem 3.5 (König, Egerváry) *If G is bipartite, then $\alpha'(G) = \beta(G)$.*

Proof: It is immediate that $\beta(G) \geq \alpha'(G)$ since for a maximum matching M, any vertex cover must contain at least one endpoint of each edge in M.

Next we shall show that $\beta(G) \leq \alpha'(G)$. Let (A, B) be a bipartition of G, let X be a vertex cover of minimum size, and define two bipartite subgraphs H_1 and H_2 so that H_1 has bipartition $(A \cap X, B \setminus X)$, H_2 has bipartition $(A \setminus X, B \cap X)$, and both H_1 and H_2 have all edges with both ends in their vertex sets.

Suppose (for a contradiction) that there does not exist a matching in H_1 which covers $A \cap X$. Then, by Hall's theorem, there is a subset $Y \subseteq A \cap X$ so that $|N_{H_1}(Y)| < |Y|$. Now, we claim that the set $X' = (X \setminus Y) \cup N_{H_1}(Y)$ is a vertex cover. Let $e \in E(G)$. If e has one end in $B \cap X$, then e is covered by X'. If e has no end in $B \cap X$, then (since X is a vertex cover) e must have one end in $A \cap X$ and the other in $B \setminus X$, so $e \in E(H_1)$. If e does not have an end in Y, then e is covered by $X \setminus Y \subseteq X'$. Otherwise, e is an edge in H_1 with one end in Y, so its other end is in $N_{H_1}(Y)$ and we again find that e is covered. But then X' is a vertex cover with $|X'| = |X| - |Y| + |N_{H_1}(Y)| < |X|$, giving us a contradiction.

Thus, H_1 has a matching M_1, which covers $A \cap X$. By a similar argument, H_2 has a matching, M_2, which covers $B \cap X$. Since these subgraphs have disjoint vertex sets, $M = M_1 \cup M_2$ is a matching of G. Furthermore, $\alpha'(G) \geq |M| = |X| = \beta(G)$. This completes the proof. $\square$

Isolated Vertex: A vertex v is *isolated* if $deg(v) = 0$. Note that if G has an isolated vertex, then G does not have an edge cover.

Theorem 3.6 (Gallai) *If G has no isolated vertex, then $\alpha'(G) + \beta'(G) = v(G)$.*

Proof: First, let M be a maximum matching (so $|M| = \alpha'(G)$). Now, we form an edge cover L from M as follows: For every vertex v not covered by M, choose an edge e incident with v and add e to L. Then L is an edge cover, so $\beta'(G) \leq |L| = |M| + v(G) - 2|M| = v(G) - \alpha'(G)$.

Next, let L be a minimum edge cover (so $|L| = \beta'(G)$) and consider the subgraph H consisting of all the vertices, and those edges in L. Since L is a minimum edge cover, it follows that $L \setminus \{e\}$ is not an edge cover for every $e \in L$. Thus, every edge $e \in E(H)$ must have one endpoint of degree 1 in H. It follows from this that every component of H is isomorphic to a star (a graph of the form $K_{1,m}$ for some positive integer m). Choose a matching $M \subseteq L$ by selecting one edge from each component of H. Then we have $\alpha'(G) \geq |M| = comp(H) = v(G) - |L| = v(G) - \beta'(G)$.

Combining the two inequalities yields $\alpha'(G) + \beta'(G) = v(G)$, as required. $\square$

Corollary 3.7 *If G is a bipartite graph without an isolated vertex, then $\alpha(G) = \beta'(G)$.*

Proof: By Observation 3.4 and Theorem 3.6 we have $\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$. Now, subtracting the relation $\beta(G) = \alpha'(G)$ proved in Theorem 3.5 we have $\alpha(G) = \beta'(G)$ as desired. $\square$

Tutte's Theorem

Odd Components: For every graph G, we let $odd(G)$ denote the number of components of G which have an odd number of vertices.

Identification: If $X \subseteq V(G)$, we may form a new graph from G by merging all vertices in X to a single new vertex. If an edge has an endpoint in X, then this edge will have the new vertex as its new endpoint. We say this graph is obtained from G by *identifying* X.

Theorem 3.8 (Tutte) *G has a perfect matching if and only if $odd(G-X) \leq |X|$ for every $X \subseteq V(G)$*

Proof: The "only if" is immediate: if G has a set $X \subseteq V(G)$ with $odd(G-X) > |X|$, then G cannot have a perfect matching.

We prove the "if" direction by induction on $v(G)$. As a base, observe that this is trivial when $v(G) \leq 2$. For the inductive step, let G be a graph for which $odd(G-X) \leq |X|$ for every $X \subseteq V(G)$ and assume the theorem holds for all graphs with fewer vertices. Call a set $X \subseteq V(G)$ *critical* if $odd(G-X) \geq |X| - 1$. We shall establish the theorem in steps.

(1) $v(G)$ is even

This follows from $odd(G-\emptyset) \leq |\emptyset| = 0$.

(2) If X is critical, then $odd(G-X) = |X|$.

This follows from the observation that $|X| + odd(G-X) \cong v(G)$ (modulo 2).

(3) There is a critical set.

For instance, $\emptyset$ is critical.

Based on (3), we may now choose a maximal critical set X. Let $|X| = k$ and let the odd components of $G-X$ be $G_1, \ldots, G_k$.

(4) $G-X$ has no even components.

If $G-X$ has an even component G', then choose $v \in V(G')$. Now $X \cup \{v\}$ is critical, contradicting the choice of X.

(5) For every $1 \leq i \leq k$ and $v \in V(G_i)$, the graph $G_i - v$ has a perfect matching.

If not, then by induction there exists $Y \subseteq V(G_i - v)$ so that $(G_i - v) - Y$ has $> |Y|$ odd components. But then $G \setminus (X \cup Y \cup \{v\})$ has $\geq |X| + |Y|$ odd components, so it is critical - again contradicting the maximality of X.

(6) G has a matching M with $|M| = k$ so that M covers X and every G_i has exactly one vertex covered by M.

Construct a graph H from G by identifying $V(G_i)$ to a new vertex y_i for every $1 \leq i \leq k$ and then deleting every loop and every edge with both ends in X. Now, H is bipartite with bipartition (X, Y) where $Y = \{y_1, \ldots, y_k\}$. Suppose (for a contradiction) that H does not have a perfect matching. Then by Hall's Theorem 3.2 there exists $Y' \subseteq Y$ with $|N_H(Y')| < |Y'|$. Let $X' = N_H(Y')$. Now the graph $G - X'$ has $\geq |Y'| > |X'|$ odd components, giving us a contradiction. So, H has a perfect matching, which proves (6).

It follows from (5) and (6) that G has a perfect matching, as desired. $\square$

Theorem 3.9 (Tutte-Berge Formula)

$$\alpha'(G) = \frac{1}{2}\left(v(G) - \max_{X \subseteq V(G)} \left(odd(G - X) - |X|\right)\right)$$

Proof: Let $k = \max_{X \subseteq V(G)}(odd(G - X) - |X|)$ and choose $X \subseteq V(G)$ so that $k = odd(G - X) - |X|$. Note that $k = odd(G - X) - |X| \cong odd(G - X) + |X| \cong v(G)$ (modulo 2). By considering X and $odd(G - X)$ we find that every matching of G must not cover $\geq k$ vertices, so $\alpha'(G) \leq \frac{1}{2}(v(G) - k)$.

To prove the other inequality, we construct a new graph G' from G by adding a set Y of k new vertices to G each adjacent to every other vertex. Let $Z' \subseteq V(G')$. We claim that $odd(G' - Z') \leq |Z'|$. If $Z' = \emptyset$, then this follows from the observation that $k \cong v(G)$ (modulo 2). If $Y \not\subseteq Z'$, then $G' - Z'$ is connected, so $|Z'| \geq 1 \geq odd(G' - Z')$. Finally, if $Y \subseteq Z'$, then we have

$$odd(G' - Z') = odd(G - Z) \leq k + |Z| = |Y| + |Z| = |Z'|.$$

Since Z' was arbitrary, Tutte's Theorem 3.8 shows that G' has a perfect matching, and it follows that G has a matching covering all but k vertices, so $\alpha'(G) \geq \frac{1}{2}(v(G) - k)$ as required. $\square$

Theorem 3.10 (Petersen) *If every vertex of G has degree 3 and G has no cut-edge, then G has a perfect matching.*

Proof: We shall show that G satisfies the condition for Tutte's Theorem. Let $X \subseteq V(G)$, let $G_1, \ldots, G_k$ be the odd components of $G - X$, and for every $1 \leq i \leq k$ let S_i be the set of edges with one end in X and the other in $V(G_i)$. Now, for every $1 \leq i \leq G_i$, we have $3v(G_i) = \sum_{v \in V(G_i)} deg_G(v) = |S_i| + 2e(G_i)$. Since $v(G_i)$ is odd, it follows that $|S_i|$ must also be odd. By our assumptions, $|S_i| \neq 1$, so we conclude that $|S_i| \geq 3$.

Now, form a new graph H from G by deleting every vertex in every even component of $G - X$, then identifying every G_i to a single new vertex y_i, and then deleting every loop and every edge with both ends in X. This graph H is bipartite with bipartition (X, Y) where $Y = \{y_1, \ldots, y_k\}$. Furthermore, by our assumptions, every vertex in Y has degree ≥ 3 and every vertex in X has degree ≤ 3. Thus, we have

$$3|X| \geq \sum_{x \in X} deg_H(x) = e(H) = \sum_{y \in Y} deg_H(y) \geq 3|Y|.$$

So, $|X| \geq |Y| = k = odd(G - X)$. Since X was arbitrary, it follows from Tutte's Theorem 3.8 that G has a perfect matching. $\square$