Midterm Solutions

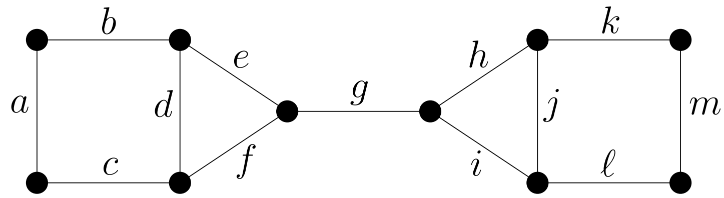
Math 345, Graph Theory I

Instructor: Matt DeVos

Name (print): ______________________________

Signature: ______________________________

Problem	Score	Value
1		3
2		3
3		6
4		6
5		8
6		4
7		10
8		4
9		6
Total:		50



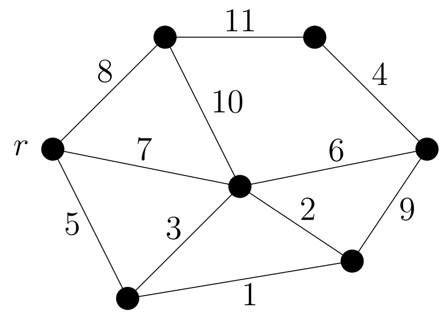
1. *(3 points)* Find a maximum matching in the above graph.

   a, d, g, j, m (other solutions are possible).

2. *(3 points)* Find a maximal matching in the above graph which is not maximum.

   a, e, h, m (other solutions are possible).

Consider the following weighted graph.



3. *(6 points)* If we use Kruskal's algorithm to choose a min cost tree, what is the sequence of edge weights we select?

1, 2, 4, 5, 6, 8

4. *(6 points)* If we use Dijkstra's algorithm to choose a shortest path tree for the vertex r in this graph, what is the sequence of edge weights we select?

5, 1, 7, 8, 6, 4

5. *(8 points)* Prove that every simple k-regular graph without a cycle of length 3 has a path of length at least $2k-1$. (hint: maximal path)

Choose a maximal path P and assume P has vertex sequence $v_1, v_2, \ldots, v_n$. By maximality, all neighbours of v_1 must be in the set $\{v_2, \ldots, v_n\}$. Furthermore, whenever v_i is a neighbour of v_1 with $i > 2$ it must be that v_{i-1} is not a neighbour of v_1 as otherwise v_1, v_i, v_{i-1} would form a triangle. It follows from this that $n \geq 2k$ so P has length at least $2k-1$ as desired.

6. *(4 points)* Find a simple k-regular graph without a triangle for which the longest path has length $2k-1$ (hint : look for a triangle-free graph with few vertices)

For every $k \geq 1$ the graph $K_{k,k}$ is simple, k-regular, and has no path of length greater than $2k-1$ (since it has only $2k$ vertices).

7. *(10 points)* Let T_1 and T_2 be spanning trees of G with $T_1 \neq T_2$. Prove that there exists $e \in E(T_1) \setminus E(T_2)$ and $f \in E(T_2) \setminus E(T_1)$ so that both $T_1 - e + f$ and $T_2 - f + e$ are spanning trees.

Choose $e \in E(T_1) \setminus E(T_2)$ and let C be the fundamental cycle of e with respect to T_2. Let u, v be the ends of e and observe that the subgraph $T_1 - e$ has exactly two components, H_1, H_2 and we may assume (without loss) that $u \in V(H_1)$ and $v \in V(H_2)$. Now, $C - e$ is a path from u to v, so there must exist an edge $f \in E(C - e)$ so that f has one end in H_1 and the other in H_2. It now follows that both $T_1 - e + f$ and $T_2 + e - f$ are spanning trees, as desired.

8. *(4 points)* Let G be a bipartite graph with bipartition (A, B) and assume that every $X \subseteq A$ satisfies $|N(X)| \geq |X|$. Define a set $X \subseteq A$ to be *tight* if $|N(X)| = |X|$. Prove that whenever $X, X' \subseteq A$ are tight then $X \cap X'$ is also tight.

The key observation is the following inequality which holds for all $X, X' \subseteq A$:

$$|N(X \cap X')| + |N(X \cup X')| \leq |N(X)| + |N(X')|$$

If X and X' are tight then using this we find

$$|N(X \cap X')| + |N(X \cup X')| \leq |N(X)| + |N(X')| = |X| + |X'| = |X \cap X'| + |X \cup X'|$$

and since $|N(X \cap X')| \geq |X \cap X'|$ and $|N(X \cup X')| \geq |X \cup X'|$ we must then have that both $X \cap X'$ and $X \cup X'$ are tight.

9. *(6 points)* Use the previous problem to give a new proof of Hall's Marriage Theorem by induction on $|A|$. (hint: let $x \in A$ and try to find a good vertex to pair with x).

If there is no tight set, then let $x \in A$, choose $y \in N(x)$ and set $G' = G - \{x, y\}$. Now for every $X \subseteq V(G')$ we have $|N_{G'}(X)| \geq |N_G(X)| - 1 \geq |X|$ so by induction, G' has a matching covering $A - \{x\}$ and then adding xy to this yields a matching covering A.

Next suppose that a tight set exists, and choose a minimal tight set X^*. Let $x \in X^*$ and let $y \in N(X^*)$ and set $G' = G - \{x, y\}$. Let $X \subseteq V(G')$ and suppose (for a contradiction) that $|N_{G'}(X)| < |X|$. We must have $|N_G(X)| = |X|$, so X is tight (in the original graph). Since $y \in N(X^*) \cap N(X)$, we must have $X \cap X^* \neq \emptyset$ (otherwise $X \cup X^*$ would violate our condition). However, then $X \cap X^*$ is tight (by the above problem) and it is a proper subset of X^* and this contradicts our choice.