3 Catalan Numbers

Catalan Numbers: The n^{th} Catalan Number is $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ for $n \geq 1$ and $C_0 = 0$

Parenthesizing: Consider a set with a nonassociative (binary) product operation. If $x_1, \ldots, x_n$ are elements in this set then we must *parenthesize* the expression $x_1 x_2 \ldots x_n$ in order to indicate the product.

Proposition 3.1 *The number of ways to parenthesize a product of n terms is C_n.*

Proof: Set u_n to be the number of ways to do this, we find the recurrence

$$u_n = \sum_{i=1}^{n-1} u_i u_{n-i}$$

for $n \geq 2$. Now the generating funciton $f(x) = \sum_{k=0}^{\infty} u_k x^k$ satisfies

$$\begin{aligned} f(x) &= x + \sum_{k=2}^{\infty} u_k x^k \\ &= x + \sum_{k=2}^{\infty} \left(\sum_{m=1}^{k-1} u_m u_{k-m} \right) x^k \\ &= x + (f(x))^2 \end{aligned}$$

Completing the square gives $\frac{1}{4} - x = (f(x) - \frac{1}{2})^2$ and now using $u_0 = 0$ we find $(\frac{1}{4} - x)^{1/2} = \frac{1}{2} - f(x)$ so $f(x) = \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2}$. Now using the binomial series (i.e. $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ where $\binom{\alpha}{k} = \frac{(\alpha)_k}{k!}$) we find

$$\begin{aligned} f(x) &= \frac{1}{2} - \frac{1}{2} \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-4)^k x^k \\ &= -\frac{1}{2} \sum_{k=1}^{\infty} \frac{(\frac{1}{2})(-\frac{1}{2}) \ldots (-k+\frac{3}{2})}{k!} (-4)^k x^k \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{(2k-3)(2k-1) \ldots (1)}{k(k-1)!} \frac{(k-1)!}{(k-1)!} (2)^k x^k \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \frac{(2k-3)(2k-1) \ldots (1)}{(k-1)!} \frac{(2k-2)(2k-4) \ldots (2)}{(k-1)!} x^k \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k-2}{k-1} x^k \qquad \square \end{aligned}$$

Up-Down Paths: We consider taking steps in $\mathbb{Z}^2$. An *Up* step takes (x, y) to $(x+1, y+1)$ while a *Down* step takes (x, y) to $(x+1, y-1)$. An *up-down path* is a sequence of Up and Down steps.

Proposition 3.2 *The number of up-down paths from* $(0, 0)$ *to* $(2n, 0)$ *which are positive except at the endpoints is* C_n.

Proof: Let $k, m > 0$ and consider up-down paths from $(0, k)$ to (n, m) which do touch the x-axis. By reflecting the segment of this walk from $(0, k)$ to the place where it first touches the x-axis about the x-axis we get an up-down walk from $(0, -k)$ to (n, m). Indeed, this establishes a correspondence between these two types of walk and we find that the number of them is $\binom{n}{(n-m-k)/2}$ (walks from $(0, -k)$ to (n, m) are just sequences of n Up or Down steps with a total of $(n-m-k)/2$ Down steps).

Now, the number of walks from $(0, k)$ to (n, m) which do not touch the x-axis is given by

$$\binom{n}{(n-m+k)/2} - \binom{n}{(n-m-k)/2}$$

So the number is equal to the number of paths from $(1, 1)$ to $(2n-1, 1)$ which do not touch the x-axis and this gives us:

$$\begin{aligned}
u_n &= \binom{2n-2}{n-1} - \binom{2n-2}{n-2} \\
&= \frac{(2n-2)!}{(n-1)!(n-1)!} - \frac{(2n-2)!}{n!(n-2)!} \\
&= \frac{n(2n-2)!}{n!(n-1)!} - \frac{(n-1)(2n-2)!}{n!(n-1)!} \\
&= \frac{1}{n}\binom{2n-2}{n-1}
\end{aligned}$$

So, the number of walks from $(0, 0)$ to $(2n, 0)$ which are positive except at the endpoints is the n^{th} Catalan number.

U-D Sequences: A *U-D sequence* of length $2n$ is a sequence $(a_1, a_2, \ldots, a_{2n})$ with the property that there are exactly n terms which are U and n which are D and for every initial subsequence $(a_1, a_2, \ldots, a_k)$ with $k \neq 1, 2n$ we have the the total number of U's is greater than the total number of D's. The following is immediate from the previous exercise.

Proposition 3.3 *The number of U-D sequences of length* $2n$ *is* C_n.

Rooted Plane Trees: A *rooted plane tree* is a tree drawn in the plane with a distinguished vertex called the *root* which has degree 1. We say that a tree is *cubic* if all vertices have degree 1 or degree 3.

Proposition 3.4

(i) *The number of rooted plane trees with* n *edges is* C_n.

(ii) *The number of rooted plane cubic trees with* $n+1$ *degree one vertices is* C_n.

Sketch of Proof: For (i), take a rooted plane tree with n edges and draw a curve just alongside the graph starting at the root, now every time we move alongside an edge bringing us further from the root we record a U and every time we move alongside an edge bringing us closer we record a D. It is easily checked that this yields a U-D sequence. Further, this construction is reversible, so it is a correspondence and this establishes (i).

For (ii) take a rooted plane cubic tree and label the degree 1 vertices other than the root $x_1, \ldots, x_n$. Then we may identify this tree with a parenthesizing of $x_1, \ldots, x_n$ by the recursive rule that if we see non root vertices of degree 1 with labels A, B which are both adjacent to the same vertex v, then we delete these two degree 1 vertices and label v by (AB). It is easily checked that this procedure produces a valid parenthesizing of $x_1, \ldots, x_n$. However, this construction is also reversible, which establishes (ii). □

Proposition 3.5 *The number of ways to triangulate a convex* n*-gon is* C_{n-1}.

Sketch of Proof: The figure shows a correspondence between rooted cubic trees and triangulations.