7 Combinatorial Geometry

Planes

Incidence Structure: An *incidence structure* is a triple $(V, B, \sim)$ so that V, B are disjoint sets and $\sim$ is a relation on $V \times B$. We call elements of V *points*, elements of B *blocks* or *lines* and we associate each line with the set of points incident with it. So, if $p \in V$ and $b \in B$ satisfy $p \sim b$ we say that p is *contained* in b and write $p \in b$ and if $b, b' \in B$ we let $b \cap b' = \{p \in P : p \sim b, p \sim b'\}$.

Levi Graph: If $(V, B, \sim)$ is an incidence structure, the associated *Levi Graph* is the bipartite graph with bipartition (V, B) and incidence given by $\sim$.

Parallel: We say that the lines b, b' are *parallel*, and write $b||b'$ if $b \cap b' = \emptyset$.

Affine Plane: An *affine plane* is an incidence structure $\mathcal{P} = (V, B, \sim)$ which satisfies the following properties:

(i) Any two distinct points are contained in exactly one line.

(ii) If $p \in V$ and $\ell \in B$ satisfy $p \notin \ell$, there is a unique line containing p and parallel to ℓ.

(iii) There exist three points not all contained in a common line.

Line: Let $\vec{u}, \vec{v} \in \mathbb{F}^n$ with $\vec{v} \neq 0$ and let $L_{\vec{u},\vec{v}} = \{\vec{u} + t\vec{v} : t \in \mathbb{F}\}$. Any set of the form $L_{\vec{u},\vec{v}}$ is called a *line* in $\mathbb{F}^n$.

AG(2, $\mathbb{F}$): We define $AG(2, \mathbb{F})$ to be the incidence structure $(V, B, \sim)$ where $V = \mathbb{F}^2$, B is the set of lines in $\mathbb{F}^2$ and if $v \in V$ and $\ell \in B$ we define $v \sim \ell$ if $v \in \ell$. It is straightforward to check that $AG(2, \mathbb{F})$ is an affine plane for every field $\mathbb{F}$. For a prime power q, we let $AG(2, q) = AG(2, \mathbb{F}_q)$.

Proposition 7.1 *If $P = (V, B, \sim)$ is an affine plane, then parallel is an equivalence relation. Further, if V is finite, there exists a natural number n called the* order *of P with the following properties:*

(i) *Every point is contained in $n + 1$ lines*

(ii) *Every line contains n points.*

(iii) *Every parallel class has size n.*

(iv) $|V| = n^2$ *and* $|B| = n^2 + n$.

Proof: Let $\ell, \ell', \ell'' \in B$ and assume $\ell || \ell'$ and $\ell' || \ell''$. If ℓ and ℓ'' intersect at the point p, then p and ℓ' contradict property (ii) of affine planes. Thus parallel is an equivalence relation. Note that it follows from axiom (ii) that every parallel class partitions V. Now, let $\mathcal{P}$ be a parallel class and let ℓ, ℓ' be lines not in $\mathcal{P}$. It then follows that ℓ and ℓ' each intersect each member of $\mathcal{P}$ in exactly one point, so ℓ and ℓ' have the same number of elements. Since axiom (iii) guarantees at least three parallel classes, any two lines must have the same number of points, and we take n to be this number. Note also that every parallel class must have size n, so $|V| = n^2$. Now, fix a point p and consider all lines through p. Any two such lines intersect at p and by the first axiom they give a partition of the remaining points. If we set t to be the number of lines through p we then find that $t(n-1) + 1 = n^2$ and from this we deduce $t = n + 1$. Since each point contains exactly one element of each parallel class, the total number of parallel classes is $n + 1$ and thus $|B| = n^2 + n$. $\square$

Projective Plane: A *projective plane* is an incidence structure $\mathcal{P} = (V, B, \sim)$ which satisfies the following properties:

(i) Any two distinct points are contained in exactly one line.

(ii) The intersection of any two lines is a single point.

(iii) Every point is nonincident with at least two lines and every line is nonincident with at least two points.

Duality: If $P = (V, B, \sim)$ is a projective plane, then $P^* = (B, V, \sim)$ is also a projective planed called the *dual* of P.

PG(2, $\mathbb{F}$): We define $PG(2, \mathbb{F})$ using the vector space $\mathbb{F}^3$. We let V be the set of all 1-dimensional subspaces of $\mathbb{F}^3$ and B to be the set of all 2-dimensional subspaces of $\mathbb{F}^3$ with the relation $v \sim b$ if $v \subseteq b$. It follows immediately from the definitions that $PG(2, \mathbb{F})$ is a projective plane. For a prime power q we define $PG(2, q) = PG(2, \mathbb{F}_q)$.

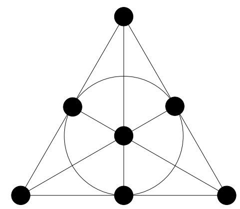

Figure 1: *The Fano Plane* PG(2,2)

Homogeneous Coordinates: If $\alpha, \beta, \gamma \in \mathbb{F}$ are not all zero, then we let $\langle \alpha, \beta, \gamma \rangle$ denote the 1-dimensional subspace spanned by (α, β, γ) and we let $[\alpha, \beta, \gamma]$ denote the 2-dimensional subspace consisting of vectors orthogonal to (α, β, γ). These are called *homogeneous coordinates* for these elements. Note that for any $\delta \in \mathbb{F} \setminus \{0\}$ we have $\langle \alpha, \beta, \gamma \rangle = \langle \delta\alpha, \delta\beta, \delta\gamma \rangle$ and $[\alpha, \beta, \gamma] = [\delta\alpha, \delta\beta, \delta\gamma]$. Further, note that a given point is in a given line if and only if their representations in homogeneous coordinates are orthogonal.

Near Pencil: Consider an incidence structure with $V = \{0, 1, \ldots, n\}$ with $B = \{[n]\} \cup \{\{0, i\} : i \in [n]\}$ and incidence defined by containment. We call any structure isomorphic to this a *near pencil.* Note that near pencils satisfy axioms (i) and (ii) for projective planes, but not (iii).

Proposition 7.2 *If $P = (V, B, \sim)$ is a finite projective plane, there exists a natural number n called the* order *of P with the following properties:*

(i) *Every point is contained in $n + 1$ lines*

(ii) *Every line contains $n + 1$ points.*

(iii) $|V| = |B| = n^2 + n + 1$

Proof: Define the *degree* of a point (line) to be the number of lines (points) it is incident with. Let $p \in V$ and $\ell \in B$ with $p \not\sim \ell$. Now, for every point $p' \in \ell$ there is a unique line containing p, p' (and any two such lines meet only at p). Similarly, for any line ℓ' containing p we have that ℓ and ℓ' intersect at a unique point in ℓ. It follows that the degree of p is equal to the degree of ℓ. We claim that every point and every line have the same degree. Were this to fail, it follows form connectivity that there must exist $V_1, V_2 \subseteq V$ and $B_1, B_2 \subseteq B$ so that $\{V_1 \cup B_1, V_2 \cup B_2\}$ is a partition of $V \cup B$ and every point in V_1 is incident with every line in

B_2 and every point in V_2 is incident with every line in B_1. It follows from (i) that there do not exist two distinct points p, p' and two distinct lines ℓ, ℓ' so that all four point-line pairs are incident. Thus, at least one of B_1, V_2 has size ≤ 1 however, this now violates (iii) giving us a contradiction. Thus, every point and line have the same degree, and we define this number to be $n+1$. For the final implication, again we consider the set of all lines through a given point p. Every such line has size $n+1$ and these lines give a partition of $V \setminus \{p\}$, so we must have $|V| = n(n+1)+1 = n^2+n+1$. By duality, we get the same result for lines. □

Proposition 7.3 *Let $P = (V, B, \sim)$ be an affine plane, and modify P to form a new incidence structure P^+ by adding a new point $\infty_{\mathcal{P}}$ for every parallel class $\mathcal{P}$, adding this new point to each line in $\mathcal{P}$, and then adding a new line incident with each newly added point. Then P^+ is a projective plane.*

Proof: It follows from axioms (i) and (ii) for affine planes that P^+ obeys axiom (i) for projective planes. It follows from our construction that P^+ obeys axiom (ii) for projective planes. It follows from axiom (iii) for affine planes that P^+ obeys axiom (iii) for projective planes. □

Proposition 7.4 *Let $P = (V, B, \sim)$ be a projective plane and let $\ell \in B$. Form a new incidence structure P^- by removing each point of ℓ, removing the line ℓ and replacing every other line ℓ' by $\ell' \setminus \ell$. Then P^- is an affine plane.*

Proof: It follows from axioms (i) and (ii) for projective planes that P^- must satisfy (i) and (ii) for affine planes. It is easy to check that every finite projective plane must have order ≥ 2, and it follows easily from this observation that P^- must satisfy axiom (iii) for affine planes. □

7.1 Desargues' Theorem

Point Perspective: We say that triangles $\triangle a_1b_1c_1$ and $\triangle a_2b_2c_2$ are *perspective from the point p* if the p, a_1a_2 are collinear, p, b_1, b_2 are collinear, and p, c_1, c_2 are collinear.

Line Perspective: We say that triangles $\triangle a_1b_1c_1$ and $\triangle a_2b_2c_2$ are *perspective from the line L* if the intersection of lines a_1b_1 and a_2b_2 lie on L and similarly for b_1c_1,b_2c_2 and c_1a_1, c_2a_2.

Theorem 7.5 (Desargues' Theorem) *If $\triangle a_1b_1c_1$ and $\triangle a_2b_2c_2$ in $PG(2,\mathbb{F})$ are perspective from a point p then they are perspective from a line.*

Proof: We may choose coordinates so that

$$\begin{aligned} a_1 &= \langle 1,0,0\rangle \\ b_1 &= \langle 0,1,0\rangle \\ c_1 &= \langle 0,0,1\rangle \end{aligned}$$

Now p must have all coordinates nonzero so without loss $p = \langle \alpha,\beta,\gamma\rangle$. Next, rescale so that

$$p = \langle 1,1,1\rangle$$

and observe that we may still denote a_1, b_1, c_1 as before. Now we may assume

$$\begin{aligned} a_2 &= \langle \alpha,1,1\rangle \\ b_2 &= \langle 1,\beta,1\rangle \\ c_2 &= \langle 1,1,\gamma\rangle \end{aligned}$$

We now have

$$\begin{aligned} \overline{a_1b_1} &= [0,0,1] & \qquad \overline{a_2b_2} &= [1-\beta, 1-\alpha, \alpha\beta-1] \\ \overline{b_1c_1} &= [1,0,0] & \qquad \overline{b_2c_2} &= [\beta\gamma-1, 1-\gamma, 1-\beta] \\ \overline{c_1a_1} &= [0,1,0] & \qquad \overline{c_2a_2} &= [1-\alpha, \alpha\gamma-1, 1-\gamma] \end{aligned}$$

It now follows that

$$\begin{aligned} \overline{a_1b_1} \cap \overline{a_2b_2} &= \langle 1-\alpha, \beta-1, 0\rangle \\ \overline{b_1c_1} \cap \overline{b_2c_2} &= \langle 0, 1-\beta, \gamma-1\rangle \\ \overline{c_1a_1} \cap \overline{c_2a_2} &= \langle \alpha-1, 0, 1-\gamma \end{aligned}$$

And we find that these three points are collinear, which completes the proof. □

Lemma 7.6 *Let $n \geq 2$, let G be an additive abelian group of order n^2 and let $H_0, H_1, \ldots, H_n \leq G$ be subgroups of order n for which $H_i \cap H_j = \{0\}$ whenever $i \neq j$. Then taking $\sim$ as containment, we have that $(G, \cup_{i=0}^n G/H_i, \sim)$ is an affine plane.*

Proof: Note that since the subgroups H_i are disjoint apart from 0 we must have

$$n^2 - 1 = |G \setminus \{0\}| \geq \sum_{i=0}^{n} |H_i \setminus \{0\}| = (n+1)(n-1) = n^2 - 1.$$

It follows from this that every nonzero element of G is in exactly one of $H_0, \dots, H_n$. Note that two elements a, b are in the same H_i coset if and only if $a - b \in H_i$. It follows from this that for distinct $a, b \in G$ there is exactly one line containing a, b, namely the coset $a + H_i$ where i is chosen so that $a - b \in H_i$. Next consider the partitions of G given by G/H_i and G/H_j where $i \neq j$. Each of these partitions the n^2 elements of G into n sets of size n, and by the above argument, no H_i coset can intersect an H_j coset in ≥ 2 points. It follows that every H_i coset meets every H_j coset in exactly one point. So, in particular, the only lines parallel to a coset $b + H_i$ are the other H_i cosets. The lemma is an immediate consequence of this. $\square$

Theorem 7.7 *There exists a projective plane which does not satisfy Desargues' theorem.*

Sketch of Proof: Let q be a power of a prime and let V be a 2-dimensional vector space over $\mathbb{F}_{q^2}$. Set $H_0, H_1, \dots, H_{q^2}$ to be the 1-dimensional subspaces of V over $\mathbb{F}_{q^2}$. We may also regard V as a 4-dimensional vector space over $\mathbb{F}_q$, and choose a 2-dimensional subspace U (over $\mathbb{F}_q$) which is not one of $H_0, \dots, H_{q^2}$. Now, U meets each of $H_0, \dots, H_{q^2}$ in a $\mathbb{F}_q$ subspace, so without loss we may assume that $|U \cap H_i| = q$ for $0 \leq i \leq q$ (so all other intersections have size 1). Now let $\{U_0, \dots, U_q\} = \{\alpha U : \alpha \in \mathbb{F}_{q^2}\}$. Then $\cup_{i=0}^{q} U_i = \cup_{i=0}^{q} H_i$. Now $U_0, \dots, U_q, H_{q+1}, \dots, H_{q^2}$ are $q^2 + 1$ abelian groups of order q^2 which partition the nonzero elements of V so we may apply the previous lemma to construct an affine plane and then complete it to a projective plane. The resulting plane fails Desargues' theorem. $\square$

7.2 Higher Geometries

There are numerous meaningful generalizations of our familiar geometric spaces; Matroids are an abstraction of the concept of linear independence, Buekenhout Geometries have different types of points and different pairwise relationships based on simple diagrams. Here we go another route: geometric lattices.

Geometric Lattice: Let $(L, \leq)$ be a lattice without infinite chains, and call the elements covering 0_L *points*. We call L a *geometric lattice* if it satisfies the following properties:

(i) L is *atomic*: every element x is the join of the points which are below x.

(ii) L is *semimodular*: If a, b are distinct and cover c then $a \vee b$ covers a and b.

In light of the assumption that L is atomic, it is natural to identify each element of L with the points below it. Then 1_L is the set of all points, $0_L = \emptyset$, and $A \wedge B = A \cap B$.

Rank: We define a *rank* function by the rule that the rank of A is the length of the longest chain from $\emptyset$ to A. It is immediate that $A < B$ implies $rk(A) < rk(B)$. The *rank* of the geometry is the rank of 1_L. Rank k elements are called *k-flats* and we call 2-flats *lines* and flats of rank $rk(1_L) - 1$ *hyperplanes*.

Proposition 7.8 *If L is a geometric lattice, p is a point in L and $A \in L$ does not contain p then $rk(p \vee A) = rk(A) + 1$.*

Proof: We proceed by induction on $rk(A)$. As a base, note that assertion is trivial when $rk(A) = 0$. For the inductive step, let $rk(A) = k \geq 1$ and choose $A' \leq A$ with $rk(A') = k-1$. Now, by induction $B = p \vee A'$ has rank k. Now by semimodularity $A \vee B$ has rank $k+1$ and since this is above both p and A, this completes the proof. $\square$

$\mathbf{AG(n}, \mathbb{F})$: Let L be the set of all affine subspaces in $\mathbb{F}^n$ (an *affine subspace* is a coset of a linear subspace) and define an order on L by inclusion. This gives a geometric lattice which we denote by $AG(n, \mathbb{F})$. We set $AG(n, q) = AG(n, \mathbb{F}_q)$.

Projective Geometry: We say that a geometric lattice L is a *projective geometry* if it satisfies the following additional properties:

(i) L is *modular*: $rk(A \vee B) + rk(A \wedge B) = rk(A) + rk(B)$ for every $A, B \in L$.

(ii) L is *connected*: There do not exist hyperplanes A, B s.t. $A \cup B$ contains every point.

$\mathbf{PG(n}, \mathbb{F})$: Let L be the set of subspaces of $\mathbb{F}^{n+1}$ and define an order on L by inclusion. This gives a projective geometry which we denote by $PG(n, \mathbb{F})$. We set $PG(n, q) = PG(n, \mathbb{F}_q)$.

Note: We have now given two definitions to $PG(2, \mathbb{F})$ which differ slightly, one is a ranked lattice with a single element of rank 0 and rank 3 and all other points of rank 1 or 2, while the other is the corresponding incidence structure on these rank 1 and rank 2 elements. We shall ignore this difference.

Closure: If S is a list of points we let $\overline{S}$ denote the join of the points in S.

Theorem 7.9 *Desargues' theorem holds in every projective geometry of rank ≥ 4.*

Proof: Let $a_1, a_2, b_1, b_2, c_1, c_2$ be points and assume that $\triangle a_1b_1c_1$ and $\triangle a_2b_2c_2$ are perspective from the point p. First suppose that the planes $P_1 = \overline{a_1b_1c_1}$ and $P_2 = \overline{a_2b_2c_2}$ are distinct and set $T = \overline{pa_1b_1c_1}$. It is immediate that $a_2, b_2, c_2 \in T$ so $P_2 \leq T$. It now follows from modularity that $P_1 \wedge P_2$ is a line L.

We claim that $\triangle a_1b_1c_1$ and $\triangle a_2b_2c_2$ are perspective from L. To see this, let $Q = \overline{pa_1b_1}$ and note that Q is a plane containing both a_2 and b_2 so it also contains the lines $\overline{a_1b_1}$ and $\overline{a_2b_2}$. Thus, these lines meet in a point q. Since q is in both P_1 and P_2 it must be that $q \in L$. Similarly, $\overline{b_1c_1}$ and $\overline{b_2c_2}$ meet at a point in L and $\overline{c_1a_1}$ and $\overline{c_2a_2}$ meet at a point in L, so we find that $\triangle a_1b_1c_1$ and $\triangle a_2b_2c_2$ are perspective from the line L.

Next suppose that $a_1, a_2, b_1, b_2, c_1, c_2$ lie in a plane P and let p_1 be a point not in P and let p_2 be another point on $\overline{pp_1}$ distinct from p, p_1. Now $\overline{p_1a_1}$ and $\overline{p_2a_2}$ are both contained in the plane $\overline{pa_1p_1}$ so they intersect at a point a^*. Similarly, $\overline{p_1b_1}$ and $\overline{p_2b_2}$ intersect at a point b^* and $\overline{p_1c_1}$ and $\overline{p_2c_2}$ intersect at a point c^*. Now, let $P^* = \overline{a^*b^*c^*}$.

Observe that P^* and P are contained in a rank 4 flat, so $P^* \cap P$ is a line L. Now $\triangle a_1b_1c_1$ and $\triangle a^*b^*c^*$ are perspective from x_1 and lie in the planes P, P^* so they are perspective from the line L. Similarly, $\triangle a_2b_2c_2$ and $\triangle a^*b^*c^*$ are perspective from L. Now $\overline{a_1b_1}$ and $\overline{a^*b^*}$ meet at a point on L and this must be the same point at which $\overline{a_2b_2}$ and $\overline{a^*b^*}$ meet, so $\overline{a_1b_1}$ and $\overline{a_2b_2}$ intersect at a point of L. Similarly, $\overline{b_1c_1}$ and $\overline{b_2c_2}$ meet at a point of L and $\overline{c_1a_1}$ and $\overline{c_2a_2}$ meet at a point of L. It follows that $\triangle a_1b_1c_1$ and $\triangle a_2b_2c_2$ are perspective from L. $\square$

Theorem 7.10 *Let P be a projective geometry.*

(i) *$P \cong PG(n, \mathbb{E})$ for a division ring $\mathbb{E}$ if and only if Desargues' Theorem holds.*

(ii) *$P \cong PG(n, \mathbb{F})$ for a field $\mathbb{F}$ if and only if it Pappus' Theorem holds*

Corollary 7.11 *Every finite projective geometry of rank ≥ 4 is isomorphic to $PG(n, \mathbb{F}_q)$.*

Proof: By Theorem 7.9 every finite projective geometry P with rank ≥ 4 satisfies Desargues' Theorem. By the previous theorem, we find that $P \cong PG(n, \mathbb{E})$ for a division ring $\mathbb{E}$. However, Witt showed that every finite division ring is a field, and this gives the desired result. $\square$