9 Latin Squares

Latin Square: A *Latin Square* of order n is an $n \times n$ array where each cell is occupied by one of n colours with the property that each colour appears exactly once in each row and in each column. We say that such an array is an $L(n)$.

$$\begin{bmatrix} \alpha & \beta & \gamma & \delta & \epsilon \\ \beta & \alpha & \epsilon & \gamma & \delta \\ \gamma & \delta & \beta & \epsilon & \alpha \\ \delta & \epsilon & \alpha & \beta & \gamma \\ \epsilon & \gamma & \delta & \alpha & \beta \end{bmatrix}$$

Cayley Table: If G is a finite group, then the multiplication table for G, also called a *Cayley table*, yields a latin square. For instance, the Cayley table of $\mathbb{Z}_4$ is the following:

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Quasigroup: A *quasigroup* is a set X equipped with a binary operation so that for every $a, b \in X$ the equations $ax = b$ and $xa = b$ have unique solutions. Equivalently, it is a latin square where the rows and columns are indexed by the set X which is also the set of colours.

Edge-Colouring: Let $K_{n,n}$ be the complete bipartite graph with bipartition (R, C) and consider an $n \times n$ array with rows indexed by R and columns indexed by C. This gives a correspondence between edges of $K_{n,n}$ and cells in this array, and we find that an n-edge-colouring of the graph corresponds to a latin square in our array.

Note: A basic fact from graph theory is that every regular bipartite graph has a perfect matching. We can easily use this to obtain an n-edge colouring of $K_{n,n}$ by choosing one perfect matching, assigning it a colour and then repeating. A deep analysis of this process shows that the number of latin squares of order n is asymptotically $e^{(1+o(1))n^2 \log n}$

Orthogonal Array: An *orthogonal array* $O(n,d)$ of *order* n and depth d is a $d \times n^2$ array where each cell contains an element of $[n]$ with the property that for any two rows, the n^2 vertical pairs of entries in these rows are distinct (so they are all possible pairs in $[n] \times [n]$).

$\mathbf{L(n) \sim OA(n,3)}$: if L is a latin square with colour set $[n]$ then the following is an $O(n,3)$:

$$\begin{bmatrix} 1 & 1 & \dots & 1 & 2 & 2 & \dots & 2 & & \dots & & n & n & \dots & n \\ 1 & 2 & \dots & n & 1 & 2 & \dots & n & & \dots & & 1 & 2 & \dots & n \\ L_{1,1} & L_{1,2} & \dots & L_{1,n} & L_{2,1} & L_{2,2} & \dots & L_{2,n} & & \dots & & L_{n,1} & L_{n,2} & \dots & L_{n,n} \end{bmatrix}$$

Conversely if A is an $O(n,3)$, then define an $n \times n$ array L where the $L_{i,j}$ is the entry in row 3 and column k where k is the unique index for which columns 1 and 2 have entries i,j. It then follows from the definition of orthogonal arrays that L is a latin square. Thus, depth three orthogonal arrays are equivalent to latin squares.

Note: In going from an $OA(n,3)$ to a $L(n)$ above we have a choice of rows. This reveals a symmetry in a latin square between the roles of rows, columns, and colours. To see this directly, let L be a latin square with rows, columns, and colour set $[n]$. Now to exchange columns and colours, we define a new $n \times n$ array where the (i,j) entry is the unique $k \in [n]$ so that row i and column k of L contain the entry j.

Transversal: A *partial transversal* of a latin square L is a selection of cells with at most one from each row, at most one from each column, and so that every colour is contained at most once. A *transversal* is a partial transversal with n cells.

Conjecture 9.1

Ryser *every latin square of odd order has a transversal.*

Brualdi *every latin square of order n has a transversal of size $n-1$.*

Note: The Cayley table for the group $\mathbb{Z}_{2n}$ has no transversal. To see this, note that the sum of all entries in any transversal is $\equiv n$ (as each element appears once). However, each entry in the table is the sum of the labels on the corresponding row and column, so the sum of all entries in the transversal is the sum of the row labels plus the sum of the column labels which is 0.

Orthogonal Latin Squares: Two latin squares L, L' with the same row, column, and colour sets are *orthogonal* if for every pair of colours (α, β) there exists a row i and a column j so that $L_{i,j} = \alpha$ and $L'_{i,j} = \beta$. Note that in this case, the set of cells where L' has a fixed colour α is a transversal of L, and more generally, such an L' exists if and only if the cells of L can be partitioned into n disjoint transversals.

Euler's 36 officer's problem: Given 36 officers with 6 ranks and 6 regiments and one soldier of each type, is is possible to arrange the soldiers in a 6×6 array so that each row and column contains exactly one soldier from each rank and each regiment? This is equivalent to the existence of two orthogonal latin squares of order 6. Euler either proved or suspected there was no solution to this problem (and knew there do not exist a pair of order 2), so he conjectured that there do not exist a pair of orthogonal latin squares of order n precisely when $n \equiv 2 \pmod 4$. This conjecture turned out false: there do not exist a pair of orthogonal latin squares of order n precisely when $n = 2, 6$.

Mutually Orthogonal Latin Squares: We say that a set $\{L^1, L^2, \ldots, L^k\}$ of latin squares with the same row, column, and colour sets are *mutually orthogonal* if any pair are orthogonal. We define $mols(n)$ to be the maximum cardinality of a set of mutually orthogonal latin squares (abbreviated MOLS) of order n.

Observation 9.2 *$mols(n) \geq k$ if and only if there exists an $OA(n, k+2)$.*

Proof: If $L^1, L^2, \ldots, L^k$ are mutually orthogonal latin squares, then the following is an $OA(n, k+2)$:

$$\begin{bmatrix} 1 & 1 & \ldots & 1 & 2 & 2 & \ldots & 2 & & \ldots & & n & n & \ldots & n \\ 1 & 2 & \ldots & n & 1 & 2 & \ldots & n & & \ldots & & 1 & 2 & \ldots & n \\ L^1_{1,1} & L^1_{1,2} & \ldots & L^1_{1,n} & L^1_{2,1} & L^1_{2,2} & \ldots & L^1_{2,n} & & \ldots & & L^1_{n,1} & L^1_{n,2} & \ldots & L^1_{n,n} \\ & & & & & & & \vdots & & & & & & & \\ L^k_{1,1} & L^k_{1,2} & \ldots & L^k_{1,n} & L^k_{2,1} & L^k_{2,2} & \ldots & L^k_{2,n} & & \ldots & & L^k_{n,1} & L^k_{n,2} & \ldots & L^k_{n,n} \end{bmatrix}$$

Conversely, if A is an $OA(n, k+2)$ then we may define k arrays $L^1, \ldots, L^k$ by the rule that $L^i(h, j)$ is the entry in row $i+2$ and column m of A where m is the unique column whose first two entries are h, j. It then follows from the definitions that $L^1, \ldots, L^k$ are mutually orthogonal latin squares. $\square$

Observation 9.3 *$mols(n) \leq n-1$ for every n.*

Proof: Let $L^1, L^2, \ldots, L^k$ be MOLS with rows, columns and colour set $[n]$. We may freely swap the columns of each L^i to arrange that the first row of each L^i is $1, 2, \ldots, n$. Now consider the entry in position $(2,1)$ of each matrix. It follows from our assumptions that all of these entries are distinct numbers in $[n] \setminus \{1\}$, so $k \leq n-1$. $\square$

Observation 9.4 *$mols(q) = q-1$ if q is a power of a prime.*

Proof: Let $a_1, a_2, \ldots a_{q-1}$ be the nonzero elements in $\mathbb{F}_q$ and for each $1 \leq k \leq q-1$ define an array L^k by the rule $L^k_{x,y} = a_k x + y$. It is immediate from this definition that each L^k is a latin square. Furthermore, if k, k' are distinct and $w, z \in \mathbb{F}$ then there exists a unique pair $(x,y) \in \mathbb{F}_q^2$ so that

$$a_k x + y = w$$
$$a_{k'} x + y = z$$

and it follows from this that L^k and $L^{k'}$ are orthogonal. $\square$

Proposition 9.5 *$mols(n) = n-1$ if and only if there exists a projective plane of order n.*

Proof: Let V be a set of n^2 points. If P_1, P_2 are partitions of V into n sets of size n, then we say that P_1 and P_2 *cross* if their common refinement is the partition of V into singletons.

Claim 1: An affine plane on V exists iff there are $n+1$ partitions of V which pairwise cross.

If there is an affine plane on V, then the parallel classes are $n+1$ partitions of V which pairwise cross. Conversely, if $P_1, \ldots, P_{n+1}$ are pairwise crossing partitions of V, then set $B = \cup_{i=1}^{n+1} P_i$. Now, for any point $x \in V$, there is exactly one member of P_i, say ℓ_i, which contains x. These sets ℓ_i must be disjoint apart from x and it follows that for every $y \in V \setminus \{x\}$ there is a unique member of B containing x and y. It follows easily from this that (V, B) is an affine plane.

Claim 2: An $OA(n, n+1)$ exists iff there are $n+1$ partitions of V which pairwise cross.

Let A be an $OA(n, n+1)$ with column set V. Then we may view the i^{th} row of A as a colouring of V using the colours $[n]$ and we let P_i be the corresponding partition of V (so two elements of V are in the same block of P_i if they have the same entry in row i). It follows immediately from the definition of orthogonal array that this gives $n+1$ pairwise crossing partitions of V. Conversely, if $P_1, \ldots P_{n+1}$ are pairwise crossing partitions of V, then we construct an $(n+1) \times n^2$ array A with columns indexed by V as follows: For each $1 \leq i \leq n+1$ we let the i^{th} row of A be an n-colouring of V so that each colour class is a block of P_i. It follows immediately from the assumption that $P_1, \ldots, P_{n+1}$ pairwise cross that the resulting array is an $OA(n, n+1)$.

Since there is an affine plane of order n if and only if there is a projective plane of order n, the two claims complete the proof. $\square$