# 4 Partitions

**Proposition 4.1** *The number of partitions of $n$ into odd parts is equal to the number of partitions of $n$ into unequal parts.*

*Proof:* The number of partitions of $n$ into unequal parts is given by the generating series $\prod_{k=1}^{\infty}(1+x^k)$ while the number of partitions of $n$ into odd parts is $\prod_{k=1}^{\infty}(1-x^{2k-1})^{-1}$. Now

$$\begin{aligned}\prod_{k=1}^{\infty}(1+x^k) &= \prod_{k=1}^{\infty}\frac{1-x^{2k}}{1-x^k} \\ &= \prod_{\ell=1}^{\infty}(1-x^{2\ell})\prod_{k=1}^{\infty}(1-x^k)^{-1} \\ &= \prod_{k=1}^{\infty}(1-x^{2k-1})^{-1}. \qquad \square\end{aligned}$$

**Partitions into Unequal Parts:** For a positive integer $k$ we let $p_e(k)$ ($p_o(k)$) denote the number of partitions of $k$ into an even (odd) number of unequal parts.

**Pentagonal Number:** We define $\omega : \mathbb{Z} \to \mathbb{Z}$ by the rule $\omega(m) = (3m^2+m)/2$. The *pentagonal numbers* are the range of $\omega$. This description is motivated by the following sequence $\omega(-1) = 1$, $\omega(-2) = 5$, $\omega(-3) = 12$.

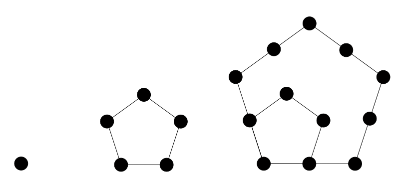

Figure 1: Some Pentagonal Numbers

**Ferrers Diagram:** The *Ferrers Diagram* associated with a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a left-aligned array of dots with $k$ rows and $\lambda_i$ dots in the $i^{th}$ row.

**Theorem 4.2 (Euler)**

$$\prod_{k=1}^{\infty}(1-x^k) = \sum_{k=0}^{\infty}(p_e(k)-p_o(k))x^k = 1+\sum_{k=1}^{\infty}(-1)^k\left(x^{\omega(k)}+x^{\omega(-k)}\right)$$

*Proof:* The first equality above is immediate. For the second, consider a Ferrers diagram of a partition of $n$ into unequal parts. Recall that the *length* of the partition is the number of rows, which we denote by $\ell$. Call the last row the *base* and let $b$ be the number of dots in the base. The *slope* is the longest line of dots starting with the last one in the first row and proceeding diagonally downward (so the corresponding line is at $45°$) and we let $s$ be the number of dots in the slope. We now define an operation on Ferrers diagrams as follows:

*Case 1:* $b \leq s$

Move the base to become the new slope unless $b = \ell = s$.

*Case 2:* $b > s$

Move the slope to become the new base unless $b - 1 = s = \ell$

It is easily verified that applying this operation twice brings us back to the same Ferrers diagram. Since this operation switches the parity of the number of parts, it follows that $p_e(n) - p_o(n)$ is zero except when there is a partition of $n$ with $b = \ell = s$ or with $b - 1 = s = \ell$ (in which case $p_e(n) - p_o(n) = (-1)^{\ell}$). In the first case $n = b + (b+1) + \ldots (b + (b-1)) = \omega(b)$ and in the second $n = (s+1) + (s+2) + \ldots (s+s) = \omega(-s)$. Combining this information yields the second inequality. $\square$