5 Partially Ordered Sets

**Poset:** A partially ordered set or poset is a pair $(P, \preceq)$ where $P$ is a set and $\preceq$ is a relation on $P$ satisfying:

(reflexive) $x \preceq x$ for every $x \in P$

(antisymmetric) $x \preceq y$ and $y \preceq x$ for $x, y \in P$ only if $x = y$

(transitive) $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ for every $x, y, z \in P$

**Graded Poset:** A ranked poset is a poset $(P, \preceq)$ together with a rank function which assigns a natural number called a rank to each element in $P$ with the property that the rank of $x$ is less than or equal to the rank of $y$ whenever $x \preceq y$.

**Inclusion Poset:** If $A, B \subseteq [n]$ we define $A \preceq B$ if $A \subseteq B$. The poset consisting of all subsets of $[n]$ with this relation is an inclusion poset. Note that giving each set $A$ the rank $|A|$ makes this a ranked poset.

**Subspace Poset:** Let $V$ be a finite vector space. If $U, U' \subseteq V$ we let $U \preceq U'$ if $U \subseteq U'$. The poset consisting of all subspaces of $V$ with this relation is a subspace poset. Note that declaring the rank of $U$ to be its dimension makes this a ranked poset.

**Divisibility Poset:** Define a relation on the positive integers by the rule that $a \preceq b$ if $a | b$. The resulting poset is the divisibility poset

**Related:** Two elements in a poset $x, y \in P$ are related if $x \preceq y$ or $y \preceq x$ and otherwise $x, y$ are unrelated.

**Maximal & Minimal:** An element $x \in P$ is maximal (minimal) if there does not exist $y \in P \setminus \{x\}$ with $x \preceq y$ ($y \preceq x$).

**Acyclic Digraph:** We shall assume that digraphs are simple (no loops or arcs with the same initial and same terminal vertices). A digraph is acyclic if it has no directed cycle.

**Transitive Closure:** If $D = (V, E)$ is an acyclic digraph, then the transitive closure of $D$ is the digraph $\bar{D}$ with vertex set $V$ and with an edge $(x, y)$ if and only if $y$ can be reached
from $x$ in $D$ by a directed path. It is easily verified that the transitive closure of an acyclic
digraph is acyclic, and satisfies $(x, y)$ and $(y, z)$ imply $(x, z)$. Thus, the relation $\preceq$ on $V$
given by the rule $x \preceq y$ if $(x, y)$ is an edge of $\overline{D}$ defines a partial order on $V$. We call this
the partial order associated with $D$.

Cover: We say that $x$ covers $y$ (in the poset $P$) if $x \neq y$ and $y \preceq x$ and there does not exist
$z \in P \setminus \{x, y\}$ such that $y \preceq z \preceq x$.

Hasse Diagram: Starting with a poset $(P, \preceq)$ we define a directed graph $H$ with vertex
set $P$ by the rule that $(x, y)$ is an edge if $y$ covers $x$ in $P$. The digraph $H$ is called a Hasse digraph for $P$ and when it is drawn in the plane with edges as straight lines going from the
lower endpoint to the upper endpoint this is called a Hasse Diagram.

Linear Order: A linear order or total order is a partial order with the property that every
pair of elements are related. In this case (assuming the set is finite) the elements may be
numbered $x_1, x_2, \ldots, x_n$ so that $x_i \leq x_j$ whenever $i \leq j$.

Linear Extension: If $(P, \preceq)$ is a poset, a linear extension of $P$ is a relation $\preceq^*$ on $P$ so
that $(P, \preceq^*)$ is a linear order and so that $x \preceq y$ implies $x \preceq^* y$.

Observation 5.1 Every finite partial order $(P, \preceq)$ has a linear extension. Further, if $x, y \in P$ are unrelated then there is a linear extension $\preceq_1$ with $x \preceq_1 y$ and a linear extension $\preceq_2$ with $y \preceq_2 x$.

Proof: We may form a linear extension inductively by simply removing a maximal element
$z$, applying induction to order the resulting poset, and then adding our $z$ back at the top.
If $x, y \in P$ are unrelated, then consider the Hasse digraph $H$. Since $x, y$ are unrelated there
are no directed paths from $x$ to $y$ or $y$ to $x$, so by adding the edge $(x, y)$ we still have an
acyclic digraph. Any linear extension $\preceq^*$ of the associated poset will have $x \preceq^* y$. □

Note: The statement that every partial order has a linear extension is called the Order Extension Principle. This statement is strictly weaker than the Axiom of Choice, but still
cannot be proved in ZF.

Conjecture 5.2 (The $\frac{1}{3} - \frac{2}{3}$ Conjecture) For every finite poset $(P, \preceq)$ which is not line-
early ordered, there exist $x, y \in P$ so that the fraction of those linear extensions for which $x$
is above $y$ is between $\frac{1}{3}$ and $\frac{2}{3}$. 

Chain: A \textit{chain} is a linearly ordered subset.

Antichain: A subset of pairwise unrelated elements.

\textbf{Theorem 5.3 (Dilworth)} If \((P, \preceq)\) is a finite poset whose maximal antichain has size \(k\), then \(P\) can be partitioned into \(k\) chains.

Intersection: The \textit{intersection} of the partial orders \((P, \preceq_1)\) and \((P, \preceq_2)\) is the order \(\preceq\) given by the rule that \(x \preceq y\) if \(x \preceq_1 y\) and \(x \preceq_2 y\). It is immediate that \((P, \preceq)\) is a partial order.

Dimension: The \textit{dimension} of a poset \((P, \preceq)\) is the minimum number of linear orders on \(P\) whose intersection is \(\preceq\).

Incidence Poset: If \(G = (V, E)\) is a graph, then the \textit{incidence poset} of \(G\) is the graded poset where every vertex has rank 1, every edge has rank 2, and for \(v \in V\) and \(e \in E\) we have \(v \preceq e\) if \(e\) and \(v\) are incident.

\textbf{Theorem 5.4 (Schnyder)} A graph \(G\) has incidence poset of dimension at most 3 if and only if \(G\) is planar.