5 Partially Ordered Sets

Poset: A *partially ordered set* or *poset* is a pair $(P, \preceq)$ where P is a set and $\preceq$ is a relation on P satisfying:

(reflexive) $x \preceq x$ for every $x \in P$

(antisymmetric) $x \preceq y$ and $y \preceq x$ for $x, y \in P$ only if $x = y$

(transitive) $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ for every $x, y, z \in P$

Graded Poset: A *ranked* poset is a poset $(P, \preceq)$ together with a *rank* function which assigns a natural number *called a rank* to each element in P with the property that the rank of x is less than or equal to the rank of y whenever $x \preceq y$.

Inclusion Poset: If $A, B \subseteq [n]$ we define $A \preceq B$ if $A \subseteq B$. The poset consisting of all subsets of $[n]$ with this relation is an *inclusion poset.* Note that giving each set A the rank $|A|$ makes this a ranked poset.

Subspace Poset: Let V be a finite vector space. If U, U' are subspaces of V we let $U \preceq U'$ if $U \subseteq U'$. The poset consisting of all subspaces of V with this relation is a *subspace poset.* Note that declaring the rank of U to be its dimension makes this a ranked poset.

Divisibility Poset: Define a relation on the positive integers by the rule that $a \preceq b$ if $a|b$. The resulting poset is the *divisibility poset*

Related: Two elements in a poset $x, y \in P$ are *related* if $x \preceq y$ or $y \preceq x$ and otherwise x, y are *unrelated.*

Maximal & Minimal: An element $x \in P$ is *maximal (minimal)* if there does not exist $y \in P \setminus \{x\}$ with $x \preceq y$ ($y \preceq x$).

Acyclic Digraph: We shall assume that digraphs are simple (no loops or arcs with the same initial and same terminal vertices). A digraph is *acyclic* if it has no directed cycle.

Transitive Closure: If $D = (V, E)$ is an acyclic digraph, then the *transitive closure* of D is the digraph $\bar{D}$ with vertex set V and with an edge (x, y) if and only if y can be reached from x in D by a directed path. It is easily verified that the transitive closure of an acyclic digraph is acyclic, and satisfies (x, y) and (y, z) imply (x, z). Thus, the relation $\preceq$ on V given by the rule $x \preceq y$ if (x, y) is an edge of $\bar{D}$ defines a partial order on V. We call this the partial order *associated with* D.

Cover: We say that x *covers* y (in the poset P) if $x \neq y$ and $y \preceq x$ and there does not exist $z \in P \setminus \{x, y\}$ such that $y \preceq z \preceq x$.

Hasse Diagram: Starting with a poset $(P, \preceq)$ we define a directed graph H with vertex set P by the rule that (x, y) is an edge if y covers x in P. The digraph H is called a *Hasse digraph* for P and when it is drawn in the plane with edges as straight lines going from the lower endpoint to the upper endpoint this is called a *Hasse Diagram*.

Linear Order: A *linear order* or *total order* is a partial order with the property that every pair of elements are related. In this case (assuming the set is finite) the elements may be numbered $x_1 x_2, \ldots, x_n$ so that $x_i \preceq x_j$ whenever $i \leq j$.

Linear Extension: If $(P, \preceq)$ is a poset, a *linear extension* of P is a relation $\preceq^*$ on P so that $(P, \preceq^*)$ is a linear order and so that $x \preceq y$ implies $x \preceq^* y$.

Observation 5.1 *Every finite partial order* $(P, \preceq)$ *has a linear extension. Further, if* $x, y \in P$ *are unrelated then there is a linear extension* $\preceq_1$ *with* $x \preceq_1 y$ *and a linear extension* $\preceq_2$ *with* $y \preceq_2 x$.

Proof: We may form a linear extension inductively by simply removing a maximal element z, applying induction to order the resulting poset, and then adding our z back at the top. If $x, y \in P$ are unrelated, then consider the Hasse digraph H. Since x, y are unrelated there are no directed paths from x to y or y to x, so by adding the edge (x, y) we still have an acyclic digraph. Any linear extension $\preceq^*$ of the associated poset will have $x \preceq^* y$. $\square$

Note: The statement that every partial order has a linear extension is called the *Order Extension Principle*. This statement is strictly weaker than the Axiom of Choice, but still cannot be proved in ZF.

Conjecture 5.2 (The $\frac{1}{3} - \frac{2}{3}$ Conjecture) *For every finite poset* $(P, \preceq)$ *which is not linearly ordered, there exist* $x, y \in P$ *so that the fraction of those linear extensions for which* x *is above* y *is between* $\frac{1}{3}$ *and* $\frac{2}{3}$.

Chain: A *chain* is a linearly ordered subset.

Antichain: A subset of pairwise unrelated elements.

Theorem 5.3 (Dilworth) *If $(P, \preceq)$ is a finite poset whose maximal antichain has size k, then P can be partitioned into k chains.*

Intersection: The *intersection* of the partial orders $(P, \preceq_1)$ and $(P, \preceq_2)$ is the order $\preceq$ given by the rule that $x \preceq y$ if $x \preceq_1 y$ and $x \preceq_2 y$. It is immediate that $(P, \preceq)$ is a partial order.

Dimension: The *dimension* of a poset $(P, \preceq)$ is the minimum number of linear orders on P whose intersection is $\preceq$.

Incidence Poset: If $G = (V, E)$ is a graph, then the *incidence poset* of G is the graded poset where every vertex has rank 1, every edge has rank 2, and for $v \in V$ and $e \in E$ we have $v \preceq e$ if e and v are incident.

Theorem 5.4 (Schnyder) *A graph G has incidence poset of dimension at most 3 if and only if G is planar.*