1 Counting and Stirling Numbers

Natural Numbers: We let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of *natural numbers*.

$[n]$: For $n \in \mathbb{N}$ we let $[n] = \{1, 2, \ldots, n\}$.

Sym: For a set X we let $Sym(X)$ denote the set of bijections from X to X.

Permutations: We define $S_n = Sym([n])$ and we call elements of S_n *permutations*. If $\pi \in S_n$ we may view π as the sequence $(\pi(1), \pi(2), \ldots, \pi(n))$

Falling Factorial: For $n, k \in \mathbb{N}$ the *falling factorial* is $(n)_k = n(n-1)(n-2)\ldots(n-k+1)$.

$\binom{\mathbf{n}}{\mathbf{k}}$: For $n, k \in \mathbb{N}$ we let $\binom{n}{k}$ denote the number of k element subsets of $[n]$.

Observation 1.1 $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Proof: Construct a k element subset $K \subseteq [n]$ by taking a permutation $\pi \in S_n$ and then letting K be the first k elements in the corresponding sequence. The total number of permutations is $n!$ and each set of size k is obtained from exactly $k!(n-k)!$ permutations since the first k and last $n-k$ elements may be freely permuted among themselves. $\square$

$\left(\binom{\mathbf{n}}{\mathbf{k}}\right)$: We let $\left(\binom{n}{k}\right)$ denote the number of multisets with ground set $[n]$ and size k.

Observation 1.2 $\left(\binom{n}{k}\right) = \binom{k+n-1}{n-1}$

Proof: Consider a sequence of length $k + n - 1$ terms each of which is either $\circ$ or $|$ and so that there are exactly k copies of $\circ$ and $n - 1$ copies of $|$. For instance $\circ| \circ \circ \circ ||\circ$. We may associate each such sequence with a multiset with ground set $[n]$ and size k by treating the number of copies of $\circ$ in between the i^{th} and $(i + 1)^{st}$ copies of $|$ as the number of copies of i in the multiset. (so the example is associated with $\{1, 2^3, 4\}$). This is a correspondence, so the total number of multisets of the given type is equal to the number of sequences, but this is just $\binom{k+n-1}{n-1}$ since we may choose any $n - 1$ of the $k + n - 1$ terms to be a $|$. $\square$

Partitions of Sets: If X is a set, a *partition of* X is a set $\mathcal{P}$ with the property that $A \cap B = \emptyset$ whenever $A, B \in \mathcal{P}$ are distinct, $\cup_{A \in \mathcal{P}} A = X$ and $\emptyset \notin X$. If $A \in \mathcal{P}$ we call A a *block* of $\mathcal{P}$.

$\mathbf{S}(n, k)$: For $n, k \in \mathbb{N}$ we let $S(n, k)$ denote the number of partitions of $[n]$ into k blocks.

Observation 1.3 $S(n,k) = kS(n-1,k) + S(n-1,k-1)$

Proof: Every partition of $[n]$ into k blocks is either obtained from a partition of $[n-1]$ into k blocks by inserting n into one of the k blocks (this can be done in k ways) or from a partition of $[n-1]$ into $k-1$ blocks by adding the new block $\{n\}$. This correspondence yields the desired equality. $\square$

Partitions of Numbers: If $n \in \mathbb{N}$ a *partition of* n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$. We say that λ is a partition of n into k parts. The Young Diagram of a partition of n is a collection of left-aligned boxes so that the number in the i^{th} row is λ_i.

$\mathbf{p}_k(n)$: For $k, n \in \mathbb{N}$ we let $p_k(n)$ denote the number of partitions of n into k parts.

Observation 1.4 $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$

Proof: The number of partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n into k parts is equal to the number of such partitions with $\lambda_k = 1$ plus the number with $\lambda_k > 1$. The first set is in correspondence with the number of partitions of $n-1$ (just remove the last element), while the second is in correspondence with the number of partitions of $n-k$ into k parts (decrease each λ_i by 1). $\square$

Indistinguishable Domain & Codomain: We say that two functions $f, g : N \to X$ are *equivalent with* N *indistinguishable* if there exists $\pi \in Sym(N)$ so that $f \circ \pi = g$ and *equivalent with* X *indistinguishable* if there exists $\sigma \in Sym(X)$ so that $\sigma \circ f = g$ (and similarly for N and X indistinguishable).

Theorem 1.5 *The following table lists the number of equivalence classes of functions from* N *to* X *where* $|N| = n$ *and* $|X| = x$ *with the indicated properties:*

Elts. of N	*Elts. of* X	*Any Function*	*Injections*	*Surjections*
dist.	*dist.*	x^n	$(x)_n$	$x!S(n,x)$
indist.	*dist.*	$\left(\binom{x}{n}\right)$	$\binom{x}{n}$	$\left(\binom{x}{n-x}\right)$
dist.	*indist.*	$S(n,1) + S(n,2)$ $\ldots + S(n,x)$	1 *if* $n \leq x$ 0 *if* $n > x$	$S(n,x)$
indist.	*indist.*	$p_1(n) + p_2(n)$ $\ldots + p_x(n)$	1 *if* $n \leq x$ 0 *if* $n > x$	$p_x(n)$

Proof: The identities for functions with N and X indistinguishable are fairly straightforward, with the last one following from the observation that every surjection $f : N \to X$ yields a partition of N as $\{f^{-1}(1), f^{-1}(2), \ldots f^{-1}(x)\}$ and each such partition gives rise to exactly $x!$ such functions. When N is indistinguishable and X is distinguishable an arbitrary function $f : N \to X$ corresponds to a multiset of size n with ground set X where the element $x \in X$ appears exactly $|f^{-1}(x)|$ times. If we add the constraint that f is injective, then we are simply counting sets instead of multisets. Finally, our surjections $f : N \to X$ correspond to multisets where every element of X appears at least once. However, by removing one copy of each element, the number of such multisets is precisely equivalent to the number of arbitrary multisets with ground set X and size $n - x$. When N is distinguishable and X is indistinguishable, we have a correspondence between partitions of N into exactly x blocks and surjections from N to X. The number of arbitrary functions from N to X is the sum of the number with range of size 1, size 2, up to size x (and so the result follows as before), and finally any two injections are equivalent so the answer for this box is 1 if such an injection exists and 0 otherwise. Finally, if both N and X are indistinguishable, then our surjections correspond precisely to partitions of the number n into x parts. The number of arbitrary functions from N to X is the sum of the number with range of size 1, size 2, up to size x (and so the result follows as before), and then any two injections are equivalent so this box is as given. $\square$

Proposition 1.6 $S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$

Proof: Let N, K be sets with $|N| = n$ and $|K| = k$. For a subset $H \subseteq K$ let $f(H)$ denote the number of functions from N to $K \setminus H$. Now using the above chart, inclusion-exclusion and the substitution $i = k - j$ we find

$$\begin{aligned} k!S(n,k) &= \#\{f : N \to K : f \text{ is a surjection}\} \\ &= \sum_{H \subseteq K} (-1)^{|H|} f(H) \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \\ &= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n \qquad \square \end{aligned}$$

Proposition 1.7 $x^n = \sum_{k=0}^{n} S(n,k)(x)_k$

Proof: The left hand side of the above equation is the total number of functions from N to X where $|N| = n$ and $|X| = x$. By counting these functions according to the size of their range we get

$$\begin{aligned} x^n &= \#\{f : N \to X\} \\ &= \sum_{k=0}^{n} \#\{f : N \to X : |f(N)| = k\} \\ &= \sum_{k=0}^{n} \binom{x}{k} k! S(n,k) \\ &= \sum_{k=0}^{n} (x)_k S(n,k) \qquad \square \end{aligned}$$

Cycles: If $f \in Sym(X)$ a *cycle of* f is a sequence $(x_1, x_2, \ldots, x_k)$ with the property that $f(x_i) = x_{i+1}$ for $1 \leq i \leq k-1$ and $f(x_k) = x_1$. We consider two cycles equivalent if one is a cyclic shift of the other. A *cycle representation* of f is a list of cycles of f including exactly one from each equivalence class.

$\mathbf{c(n,k)}$: For $n, k \in \mathbb{N}$ we let $c(n,k)$ denote the number of permutations of $[n]$ with exactly k cycles. Note that $c(0,0) = 1$ but $c(s,0) = c(0,t)$ whenever $s, t > 0$.

Observation 1.8 $c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$

Proof: Every permutation of $[n]$ with k cycles is either obtained from a permutation of $[n-1]$ with k cycles by inserting n into any of the $n-1$ positions immediately following some number (which can be done in $n-1$ ways) or from a permutation of $[n-1]$ with $k-1$ cycles by adding a new cycle (n). This correspondence gives the above equation. $\square$

$\mathbf{s(n,k)}$: We define the *Stirling number of the first kind* by $s(n,k) = (-1)^{n-k}c(n,k)$

Proposition 1.9

(i) $\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)\ldots(x+n-1)$

(ii) $(x)_n = \sum_{k=0}^{n} s(n,k)x^k$

Proof: For (i) we shall consider the left hand side and the right hand side as polynomials in x. Let $F_n(x)$ denote the right hand side and define the coefficients $b(n,k)$ by the rule $F_n(x) = \sum_{k=0}^{n} b(n,k)x^k$ where $b(0,0) = 1$ and $b(s,0) = b(0,t) = 0$ whenever $s,t > 0$. Now we have

$$\begin{aligned}\sum_{k=0}^{n} b(n,k)x^k &= F_n(x)\\ &= (x+n-1)F_{n-1}(x)\\ &= \sum_{k=0}^{n-1} b(n-1,k)x^{k+1} + (n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k\\ &= \sum_{k=1}^{n} b(n-1,k-1)x^k + (n-1)\sum_{k=0}^{n-1} b(n-1,k)x^k\end{aligned}$$

So we find that $b(n,k) = b(n-1,k-1) + (n-1)b(n-1,k)$ for $n,k \geq 1$. It follows that the terms $b(n,k)$ satisfy the same recurrence as $c(n,k)$ and are equal whenever either input is zero, so we find that $b(n,k) = c(n,k)$. This completes the proof of (i).

For (ii) we have

$$\begin{aligned}\sum_{k=0}^{n} s(n,k)x^k &= \sum_{k=0}^{n}(-1)^{n-k}c(n,k)x^k\\ &= \sum_{k=0}^{n}(-1)^{n}c(n,k)(-x)^k\\ &= (-1)^n \cdot (-x)(-x+1)(-x+2)\dots(-x+n-1)\\ &= x(x-1)(x-2)\dots(x-n+1)\\ &= (x)_n \qquad \square\end{aligned}$$

$\mathbb{F}[\mathbf{x}]$: For a field $\mathbb{F}$ we let $\mathbb{F}[x]$ denote the ring of polynomials with indeterminate x and coefficients in $\mathbb{F}$.

Bases of $\mathbb{C}[x]$: We define B_1 to be the basis of $\mathbb{C}[x]$ given by $B_1 = \{1, x, x^2, x,^3, \dots\}$ and B_2 to be the basis of $\mathbb{C}[x]$ given by $\{1, (x), (x)_2, (x)_3, \dots\}$.

Proposition 1.10 *Regard* $s = \{s(n,k)\}_{n,k\in\mathbb{N}}$ *and* $S = \{S(n,k)\}_{n,k\in\mathbb{N}}$ *as infinite matrices. Then we have:*

(i) S *is the basis transformation matrix from* B_2 *and* B_1*.*

(ii) s *is the basis transformation matrix from* B_1 *to* B_2*.*

(iii) S *and* s *are inverse matrices.*

(iv) $\sum_{k=m}^{n} S(n,k)s(k,m) = \delta_{mn}$

Proof: Parts (i) and (ii) follow immediately from Propositions 1.6 and 1.9. Part (iii) is an immediate consequence of (i) and (ii), and (iv) is a restatement of (iii). $\square$