2 Points, Vectors, and Lines

N-Space

Notation. Our main setting for this class is $\mathbb{R}^n$. We will always use boldface to denote an element of this set, say $\mathbf{x} \in \mathbb{R}^n$, and we will most commonly write $\mathbf{x} = (x_1, \ldots, x_n)$ to indicate the n real number coordinates that comprise $\mathbf{x}$. However, we will sometimes instead use the notation $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ when this is more convenient (usually when we are doing some matrix operations).

Points or Vectors? An element $\mathbf{x} \in \mathbb{R}^n$ may be viewed **both** as a point in space **and** as a "direction". We will generally call $\mathbf{x}$ a *point* to emphasize the first context or a *vector* to emphasize the second. However, it is vital to understand that every $\mathbf{x} \in \mathbb{R}^n$ may be thought of both ways.

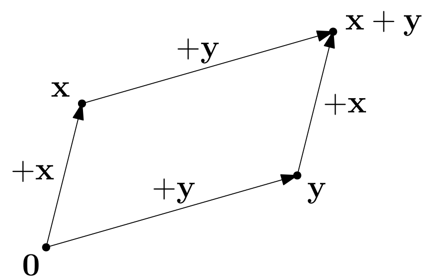
Zero We define $\mathbf{0} = (0, \ldots, 0)$. One may view $\mathbf{0}$ both as the origin and as a vector indicating the trivial direction.

Sums If $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, the sum of $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} + \mathbf{y} = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

Addition is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ and associative: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

Note: In figures we give points labels of the form $\mathbf{x}$ (as usual). We will label an arrow $+\mathbf{y}$ to indicate that adding $\mathbf{y}$ takes you from the point at the start of the arrow to the point at the end. For instance, the following figure demonstrates the commutativity of the sum.



Scalar Multiplication If $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$, the scalar product of t and $\mathbf{x}$ is defined to be

$$t\mathbf{x} = t(x_1, \ldots, x_n) = (tx_1, \ldots, tx_n).$$

Note: Assuming $\mathbf{x} \in \mathbb{R}^n$ is nonzero (i.e. $\mathbf{x} \neq \mathbf{0}$) the line through $\mathbf{0}$ and $\mathbf{x}$ is precisely the set of all scalar multiples of $\mathbf{x}$.

Example:

Lines

If L is a line through $\mathbf{0}$ and $\mathbf{x}$ is any point on L except $\mathbf{0}$, then we have $L = \{t\mathbf{x} \mid t \in \mathbb{R}\}$. How can we describe a line that does not go through $\mathbf{0}$?

Observation 2.1. *If L is a line in $\mathbb{R}^n$, then for any point $\mathbf{w} \in L$ and any vector $\mathbf{x}$ parallel to (or in the same direction as) L we have*

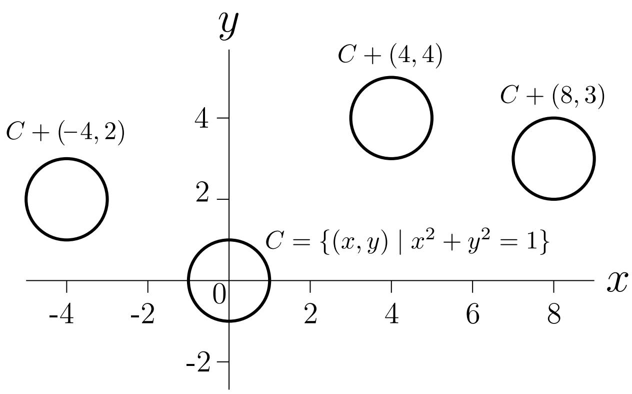
$$L = \{\mathbf{w} + t\mathbf{x} \mid t \in \mathbb{R}\}.$$

Next we will introduce a little more notation to think about this in another way.

Definition. If $S \subseteq \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$, the translate of S by $\mathbf{x}$ is defined to be

$$\mathbf{x} + S = \{\mathbf{x} + \mathbf{s} \mid \mathbf{s} \in S\}$$

Example.

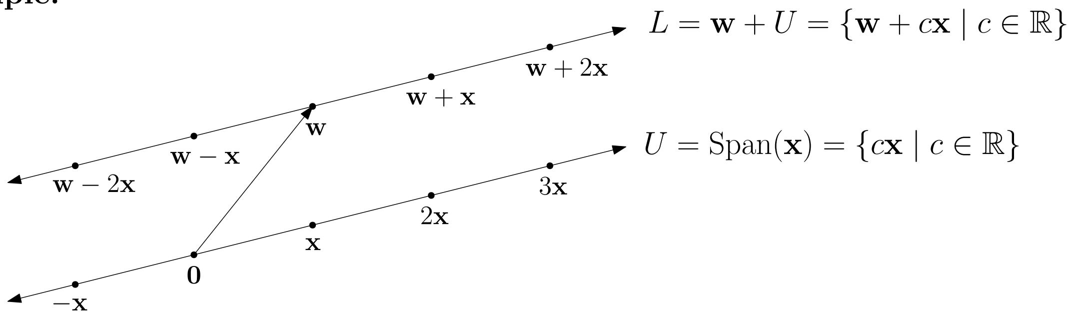


Note: With this framework, we have another way of describing a line, namely

$$L = \{\mathbf{w} + t\mathbf{x} \mid t \in \mathbb{R}\} = \mathbf{w} + \{t\mathbf{x} \mid t \in \mathbb{R}\}.$$

So the line in the direction of $\mathbf{x}$ going through the point $\mathbf{w}$ can also be thought of as a translate of the line in the direction of $\mathbf{x}$ going through $\mathbf{0}$.

Example.



Linear Combinations

Definition: If $\mathbf{x_1}, \ldots, \mathbf{x_k} \in \mathbb{R}^n$ and $c_1, \ldots, c_k \in \mathbb{R}$ then we call

$$c_1\mathbf{x_1} + \ldots + c_k\mathbf{x_k}$$

a *linear combination* of $\mathbf{x_1}, \ldots, \mathbf{x_k}$. The set of all linear combinations of $\mathbf{x_1}, \ldots, \mathbf{x_k}$ is called the *span* of $\mathbf{x_1}, \ldots, \mathbf{x_k}$ and we express by

$$\mathrm{Span}(\mathbf{x_1}, \ldots, \mathbf{x_k}) = \{c_1\mathbf{x_1} + \ldots + c_k\mathbf{x_k} \mid c_1, \ldots, c_k \in \mathbb{R}\}.$$

Example: For a single nonzero vector $\mathbf{x}$ we have

$$\mathrm{Span}(\mathbf{x}) = \{t\mathbf{x} \mid t \in \mathbb{R}\}$$

So the span of $\mathbf{x}$ is a line through $\mathbf{0}$.

Example: Suppose (for the sake of concreteness) that we are in $\mathbb{R}^3$. If $\mathbf{x_1}$ and $\mathbf{x_2}$ are

Definition. A set $U \subseteq \mathbb{R}^n$ is called a *subspace* if it satisfies the following three properties:

zero $\mathbf{0} \in U$.

additive closure If $\mathbf{x}, \mathbf{y} \in U$ then $\mathbf{x} + \mathbf{y} \in U$.

scalar mult. closure If $\mathbf{x} \in U$ and $t \in \mathbb{R}$ then $t\mathbf{x} \in U$.

Theorem 2.2. *A set* $U \subseteq \mathbb{R}^n$ *is a subspace if and only if* $U = \mathrm{span}(\mathbf{x_1}, \ldots, \mathbf{x_k})$ *holds for some list of vectors.*

Definition: If $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$, then we define

$$\mathrm{Int}(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \{t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \ldots + t_k\mathbf{x}_k \mid \text{ where } t_1, \ldots, t_k \in \mathbb{Z}\ \}$$

If $\Lambda = \mathrm{Int}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ and Λ does not contain arbitrarily small nonzero vectors, we call Λ a lattice.

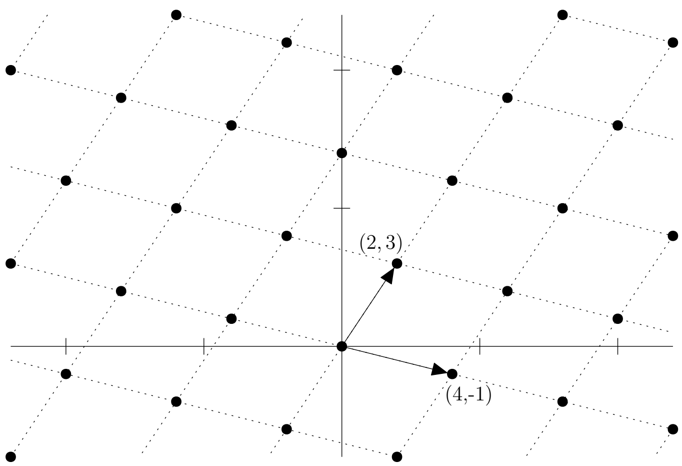


Figure 1: *
The lattice $\Lambda((2,3),(4,-1))$