3 Distances and Dot Products

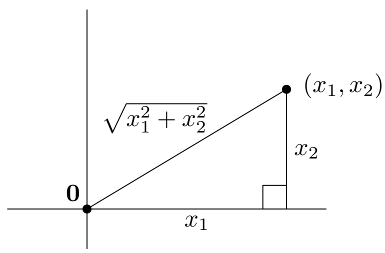
Norms and Distance

Definition: We define the *norm* of $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ to be

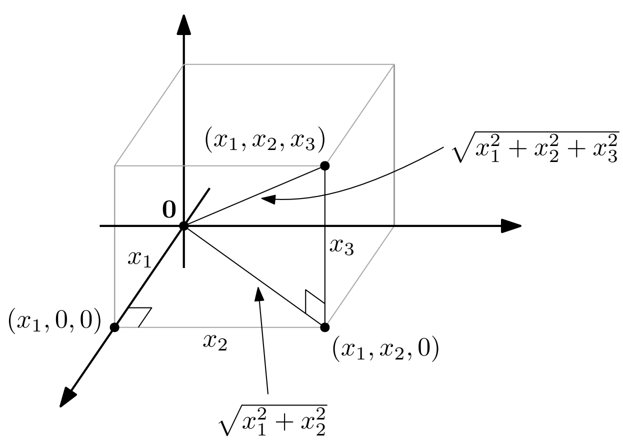
$$||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

Lemma 3.1. *For every point $\mathbf{x} \in \mathbb{R}^n$, the distance between $\mathbf{0}$ and $\mathbf{x}$ is $||\mathbf{x}||$.*

Proof. If $n = 1$ then $\mathbf{x} = (x_1)$ and $||\mathbf{x}|| = |x_1|$ is the distance between the origin and $\mathbf{x}$. For $n = 2$ this is a familiar consequence of the Pythagorean theorem as shown in the figure below.



For $n \geq 3$ we deduce the result by using two applications of the Pythagorean Theorem (see the following figure). First apply Pythagoras to the triangle with vertices $(0, 0, 0)$, $(x_1, 0, 0)$ and $(x_1, x_2, 0)$ to deduce that the distance between $(x_1, x_2, 0)$ and the origin is $\sqrt{x_1^2 + x_2^2}$. Then apply Pythagoras to the triangle with vertices $(0, 0, 0)$, $(x_1, x_2, 0)$, and (x_1, x_2, x_3) to deduce that the distance between (x_1, x_2, x_3) and 0 is $\sqrt{x_1^2 + x_2^2 + x_3^2}$ as desired.



The general case follows by a similar argument. Using the Pythagorean Theorem to the triangle with vertices $(0,0,\ldots,0)$ and $(x_1,0,\ldots,0)$ and $(x_1,x_2,\ldots 0)$ we deduce that the distance between $(0,0,\ldots 0)$ and $(x_1,x_2,0,0,\ldots 0)$ is $\sqrt{x_1^2+x_2^2}$. Then, using the Pythagorean Theorem to the triangle with vertices $(0,0,\ldots,0)$ and $(x_1,x_2,0,\ldots 0)$ and $(x_1,x_2,x_3,0,\ldots 0)$ we deduce that the distance between $(0,0,\ldots 0)$ and $(x_1,x_2,x_3,0,\ldots 0)$ is $\sqrt{x_1^2+x_2^2+x_3^2}$. Continuing in this manner we eventually find that the distance between $(0,0,\ldots 0)$ and $(x_1,x_2,\ldots,x_n)$ is $\sqrt{x_1^2+x_2^2+\ldots+x_n^2}$ as claimed. □

Theorem 3.2. *For any two points* $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$*, the distance between* $\mathbf{x}$ *and* $\mathbf{y}$ *is*

$$\mathrm{dist}(\mathbf{x},\mathbf{y}) = ||\mathbf{x}-\mathbf{y}||$$

Dot Products

Definition: If $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)$, the *dot product* of $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n.$$

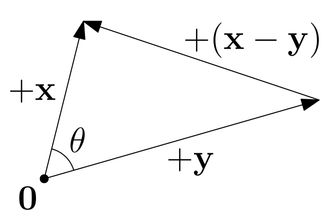
Notes:

- The dot product is commutative: $\mathbf{x}\cdot\mathbf{y}=\mathbf{y}\cdot\mathbf{x}$
- If $t\in\mathbb{R}$ then $(t\mathbf{x})\cdot\mathbf{y}=t(\mathbf{x}\cdot\mathbf{y})$
- The dot product obeys $\mathbf{x}\cdot(\mathbf{y}+\mathbf{z})=\mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}$.
- $\mathbf{x}\cdot\mathbf{x}=||\mathbf{x}||^2$

Proposition 3.3 (Dot product formula)**.** *If* $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ *make an angle of* θ*, then*

$$\mathbf{x}\cdot\mathbf{y} = ||\mathbf{x}||\,||\mathbf{y}||\cos\theta.$$

Proof. Assume that $\mathbf{x}$ and $\mathbf{y}$ span a 2-dimensional subspace as shown in the figure below. (The other case will be homework!)



Using the Law of Cosines and some elementary properties of the dot product we have

$$\begin{aligned}||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||\,||\mathbf{y}||\cos\theta &= ||\mathbf{x}-\mathbf{y}||^2 \\ &= (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) \\ &= \mathbf{x}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} - 2(\mathbf{x}\cdot\mathbf{y}) \\ &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2(\mathbf{x}\cdot\mathbf{y})\end{aligned}$$

and this equation immediately simplifies to the desired identity. □

Corollary 3.4. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *make an angle of* θ *where* $0 \le \theta \le \pi$*, then*

$$\mathbf{x}\cdot\mathbf{y}\begin{cases} > 0 & \textit{if } \theta < \frac{\pi}{2} \quad (\theta \textit{ is acute}) \\ = 0 & \textit{if } \theta = \frac{\pi}{2} \quad (\theta \textit{ is a right angle}) \\ < 0 & \textit{if } \theta > \frac{\pi}{2} \quad (\theta \textit{ is obtuse}) \end{cases}$$

Definition: Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are *orthogonal* if $\mathbf{x}\cdot\mathbf{y} = 0$. Note: if $\mathbf{x}$ and $\mathbf{y}$ are nonzero, they are orthogonal if and only if they make an angle of $\frac{\pi}{2}$