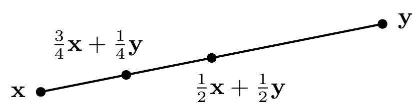
7 Convexity

Line Segments. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the midpoint between $\mathbf{x}$ and $\mathbf{y}$ is given by the equation $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$. This is apparent since each coordinate of the midpoint is the average of the corresponding coordinates of $\mathbf{x}$ and $\mathbf{y}$. More generally, the line segment between $\mathbf{x}$ and $\mathbf{y}$, denoted $\overline{\mathbf{xy}}$, is given by

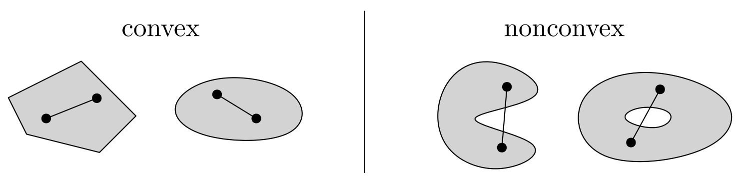
$$\overline{\mathbf{xy}} = \{t\mathbf{x} + (1-t)\mathbf{y} \mid 0 \leq t \leq 1\}.$$

Example:



Definition. A subset $S \subseteq \mathbb{R}^n$ is called *convex* if for every $\mathbf{x}, \mathbf{y} \in S$ the entire line segment $\overline{\mathbf{xy}}$ is contained in S.

Example:



Observation 7.1. *Every affine set is convex.*

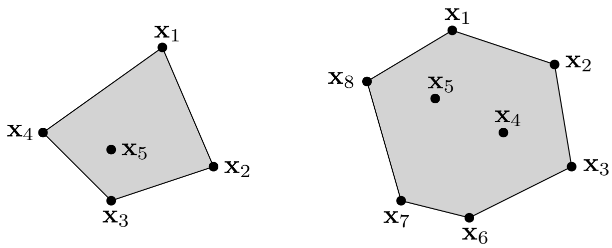
Definition. If $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$, a linear combination

$$c_1\mathbf{x_1} + \ldots + c_k\mathbf{x_k}$$

is called a *convex combination* if $c_1, \ldots, c_k \geq 0$ and $c_1 + \ldots + c_k = 1$. We define the *convex hull* of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ to be the set of all convex combinations of these points, denoted

$$\mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \{c_1\mathbf{x_1} + \ldots + c_k\mathbf{x_k} \mid c_1, \ldots, c_k \geq 0 \text{ and } c_1 + \ldots + c_k = 1\}$$

Example:



Lemma 7.2. *For every* $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^n$ *the set* $\mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ *is convex.*

Proof. Let $\mathbf{y}, \mathbf{z} \in \mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ and let $0 \leq t \leq 1$. To prove that $\mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ is convex it suffices to show that $t\mathbf{y} + (1-t)\mathbf{z} \in \mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$. To achieve this, use the fact that $\mathbf{y}, \mathbf{z} \in \mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ to express these points as

$$\mathbf{y} = c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k \text{ and}$$
$$\mathbf{z} = d_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k.$$

where $c_1, \ldots, c_k, d_1, \ldots, d_k \geq 0$ and $c_1 + \ldots + c_k = 1$ and $d_1 + \ldots + d_k = 1$. Now, the point of interest is

$$t\mathbf{y} + (1-t)\mathbf{z} = (tc_1 + (1-t)d_1)\mathbf{x}_1 + \ldots + (tc_k + (1-t)d_k)\mathbf{x}_k.$$

All of the coefficients of the $\mathbf{x}_i$ on the right in the above equation are nonnegative, and $\left(t\sum_{i=1}^k c_i\right) + \left((1-t)\sum_{i=1}^k d_i\right) = t + (1-t) = 1$ and it follows that the point $t\mathbf{y} + (1-t)\mathbf{z}$ is in $\mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ as desired. $\square$

Theorem 7.3. *The set* $\mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ *is the unique minimal convex set containing the points* $\mathbf{x}_1, \ldots, \mathbf{x}_k$*.*

Proof. To prove this theorem, it suffices to show that every convex set S containing $\mathbf{x}_1, \ldots, \mathbf{x}_k$ satisfies $\mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k) \subseteq S$. We prove this by induction on k. The base case $k = 1$ follows from the observation that $\mathrm{Conv}(\mathbf{x_1}) = \{\mathbf{x_1}\}$, so if the set S contains $\mathbf{x_1}$ then $\mathrm{Conv}(\mathbf{x_1}) \subseteq S$. For the inductive step we have $k \geq 2$ and the inductive hypothesis tells us

$$\mathrm{Conv}(\mathbf{x_1}, \ldots, \mathbf{x_{k-1}}) \subseteq S \tag{1}$$

To prove that $\mathrm{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k) \subseteq S$ we let $\mathbf{y}$ be an arbitrary point in $\mathrm{Conv}(\mathbf{x_1}, \ldots, \mathbf{x_k})$ and we will show $\mathbf{y} \in S$. By the definition of the convex hull we may choose $c_1, \ldots, c_k \geq 0$ with $c_1 + \ldots + c_k = 1$ so that

$$\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_k\mathbf{x}_k.$$

If $c_k = 1$ then $c_1, \ldots, c_{k-1} = 0$ so $\mathbf{y} = \mathbf{x}_k \in S$ and the proof is complete. So, we may assume $c_k < 1$ and we may then define coefficients $c'_1, \ldots, c'_{k-1}$ by the rule

$$c'_i = \frac{c_i}{1 - c_k}.$$

Note that $c'_1 + \ldots + c'_{k-1} = \frac{1}{1-c_k}(c_1 + \ldots + c_{k-1}) = \frac{1}{1-c_k}(1 - c_k) = 1$. Now define

$$\mathbf{y}' = c'_1\mathbf{x}_1 + \ldots + c'_{k-1}\mathbf{x}_{k-1}$$

We have now shown that $\mathbf{y}' \in \mathrm{Conv}(\mathbf{x_1}, \ldots, \mathbf{x_{k-1}})$ so by equation (1) we have $\mathbf{y}' \in S$. Now using the fact that S is convex and $\mathbf{y}', \mathbf{x_k} \in S$ we deduce that $(1-c_k)\mathbf{y}' + c_k\mathbf{x}_k \in S$. However, $(1 - c_k)\mathbf{y}' + c_k\mathbf{x}_k = c_1\mathbf{x}_1 + \ldots + c_k\mathbf{x}_k = \mathbf{y}$ so this completes the proof. $\square$