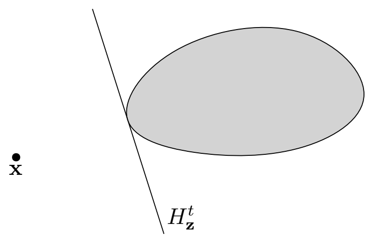
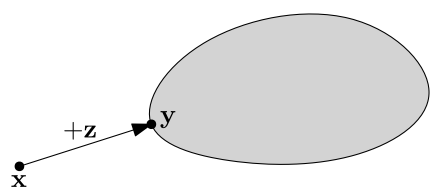
8 Separating Hyperplanes

Definition. A set $S \subseteq \mathbb{R}^n$ is *closed* if it has the property that whenever a sequence of points in S, say $\mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to a point $\mathbf{x}$, then we also have $\mathbf{x} \in S$.

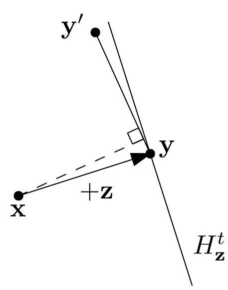
Proposition 8.1. *Let $S \subseteq \mathbb{R}^n$ be a closed and convex set and let $\mathbf{x} \in \mathbb{R}^n$ satisfy $\mathbf{x} \notin S$. Then there exists a hyperplane $H^t_{\mathbf{z}}$ separating $\mathbf{x}$ and S: Namely $\mathbf{x} \cdot \mathbf{z} < t$ and every $\mathbf{w} \in S$ satisfies $\mathbf{w} \cdot \mathbf{z} \geq t$.*



Proof. Let $\mathbf{y} \in S$ be a closest point to $\mathbf{x}$ and define the (nonzero) vector $\mathbf{z} = \mathbf{y} - \mathbf{x}$ and the scalar $t = \mathbf{z} \cdot \mathbf{y}$.



Note that $\mathbf{z} \cdot \mathbf{x} = \mathbf{z} \cdot (\mathbf{y} - \mathbf{z}) = t - \mathbf{z} \cdot \mathbf{z} < t$. So, the point $\mathbf{y}$ lies on the hyperplane $H^t_{\mathbf{z}}$ and the point $\mathbf{x}$ lies on the side of this hyperplane containing those points whose dot product with $\mathbf{z}$ is less than t. Suppose (for a contradiction) that there exists $\mathbf{y}' \in S$ so that $\mathbf{z} \cdot \mathbf{y}' < t$. In this case, there is a plane in $\mathbb{R}^n$ containing $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{y}'$ as in the figure below.



However, this picture is contradictory. Since S is convex, the line segment $\overline{\mathbf{y}\mathbf{y}'}$ is contained in S. But this segment contains a point closer to $\mathbf{x}$ than $\mathbf{y}$, and contradicts the choice of $\mathbf{y}$. It follows that every $\mathbf{w} \in S$ satisfies $\mathbf{w} \cdot \mathbf{z} \geq t$ as desired. □