9 Polytopes

Definition. A *polytope* is a subset of $\mathbb{R}^n$ of the form $P = \text{Conv}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$. The *dimension* of P is the dimension of $\text{Aff}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$. 2-dimensional polytopes are also known as convex polygons, while 3-dimensional polytopes are known as polyhedra.

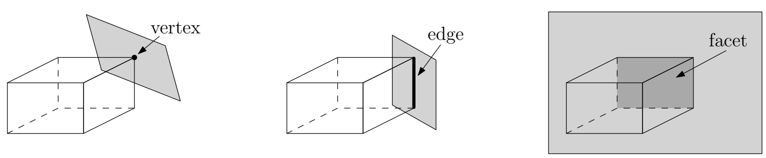
Examples:

polygons polyhedra

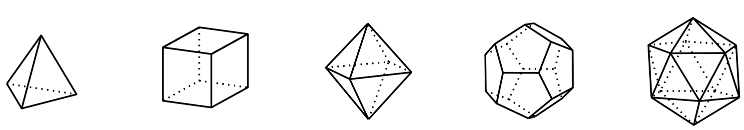
Definition. Let $\mathbf{y} \in \mathbb{R}^n$ and let $t \in \mathbb{R}$. We define the *half-space*

$$H_{\mathbf{y}}^{\leq t} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{y} \cdot \mathbf{x} \leq t\}.$$

Definition. Let P be a polytope of dimension d and let $H_{\mathbf{y}}^{\leq t}$ be a half-space with $P \subseteq H_{\mathbf{y}}^{\leq t}$. If $F = H_{\mathbf{y}}^{t} \cap P \neq \emptyset$ then we call F a *face* of P. Every face of P is another polytope, so it has a dimension. A face of dimension 0 consists of a single point and is called a *vertex*. A face of dimension 1 is called an *edge*, and a face of dimension $d-1$ is called a *facet*.



Definition. We define *regular polytopes* recursively. Every 1-dimensional polytope is regular. A polytope P of dimension $d > 1$ is regular if every facet is a copy of the same regular $(d-1)$-dimensional polytope and for any two vertices u, v there is a symmetry of P which takes u to v. The regular polytopes of dimension 3 are known as the *Platonic Solids*, they are as follows:



Tetrahedron Cube Octahedron Dodecahedron Icosahedron