21 Products of Mirrors

We have established a theory of isometries of $\mathbb{R}^n$ by way of linear algebra. However our treatment of this subject is not at all similar to the manner in which it was discovered. Indeed, isometries of the plane were understood long before the modern linear algebra tools we have been using were developed. In this section we will return to thinking about isometries, but work with them in terms of products of mirrors. This provides an alternate perspective on the theory.

Fixed Points

Definition. For any transformation $F \in \mathrm{Trans}(X)$ we let $\mathrm{Fix}(F)$ denote the set of all fixed points of F. So $\mathrm{Fix}(F) = \{x \in X \mid F(x) = x\}$.

Lemma 21.1. *Let $F \in AGL_n$.*

1. *If $\mathbf{y_1}, \ldots, \mathbf{y_k} \in \mathrm{Fix}(F)$, then every point in $\mathrm{Aff}(\mathbf{y_1}, \ldots, \mathbf{y_k})$ is in $\mathrm{Fix}(F)$.*
2. *$\mathrm{Fix}(F)$ must be affine.*

Proof. To prove the first part, let F be given by the rule $F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ and let $\mathbf{y} \in \mathrm{Aff}(\mathbf{y_1}, \ldots, \mathbf{y_k})$. Since $\mathbf{y}$ is in the affine hull of $\mathbf{y_1}, \ldots, \mathbf{y_k}$ we may choose $c_1, \ldots, c_k \in \mathbb{R}$ so that $\mathbf{y} = c_1\mathbf{y_1} + \ldots + c_k\mathbf{y_k}$ and $c_1 + \ldots + c_k = 1$. Now we have

$$\begin{aligned} F(\mathbf{y}) &= A\mathbf{y} + \mathbf{b} \\ &= A(c_1\mathbf{y_1} + \ldots + c_k\mathbf{y_k}) + (c_1 + \ldots + c_k)\mathbf{b} \\ &= c_1(A\mathbf{y_1} + \mathbf{b}) + \ldots + c_k(A\mathbf{y_k} + \mathbf{b}) \\ &= c_1\mathbf{y_1} + \ldots + c_k\mathbf{y_k} \\ &= \mathbf{y} \end{aligned}$$

□

The second part follows immediately from the first.

Note: The fact that $\mathrm{Fix}(F)$ is always affine generalizes a property we already saw for isometries of the plane: Every isometry of the plane must either fix the entire plane (the identity), a line (a mirror), a point (a nontrivial rotation), or nothing (nontrivial translations and glides).

Definition. Our concept of a mirror works in higher dimensions too. If H is a hyperplane in $\mathbb{R}^n$, say $H = H^t_{\mathbf{y}}$ then there is an isometry M_H given by the rule that every point $\mathbf{x}$ in the hyperplane stays fixed, and every point $\mathbf{x}$ not in H maps to the unique point on the line $\mathbf{x} + \mathrm{Span}(\mathbf{y})$ that is the same distance from H as $\mathbf{x}$. Although we will not supply a proof here, this function M_H is an isometry for every hyperplane H.

Lemma 21.2. *If F is a non-identity isometry, there is a hyperplane H so that* $\mathrm{Fix}(F) \subset \mathrm{Fix}(M_H F)$.

Proof. Since F is not the identity, we may choose $\mathbf{x} \in \mathbb{R}^n$ so that $\mathbf{y} = F(\mathbf{x})$ satisfies $\mathbf{y} \neq \mathbf{x}$. Let H be the hyperplane orthogonal to the vector $\mathbf{y} - \mathbf{x}$ that contains the point $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$. This hyperplane H can also be described as follows:

$$H = \{\mathbf{z} \in \mathbb{R}^n \mid \mathrm{dist}(\mathbf{x}, \mathbf{z}) = \mathrm{dist}(\mathbf{y}, \mathbf{z})\}.$$

Now, let $\mathbf{z}$ be a fixed point of F. Since F is an isometry we must have

$$\mathrm{dist}(\mathbf{x}, \mathbf{z}) = \mathrm{dist}(F(\mathbf{x}), F(\mathbf{z})) = \mathrm{dist}(\mathbf{y}, \mathbf{z}).$$

Therefore, every fixed point of F lies on the hyperplane H. So every fixed point of F is also a fixed point of M_H. Now we claim that the function $M_H F$ satisfies the lemma. Every fixed point $\mathbf{z}$ of F satisfies $M_H F(\mathbf{z}) = M_H(\mathbf{z}) = \mathbf{z}$ so we have $\mathrm{Fix}(F) \subseteq \mathrm{Fix}(M_H F)$. However the point $\mathbf{x}$ is not a fixed point of F but $M_H F(\mathbf{x}) = M_H(\mathbf{y}) = \mathbf{x}$, so $\mathbf{x}$ is a fixed point of $M_H F$. This completes the proof. $\square$

Proposition 21.3. *Every isometry of $\mathbb{R}^n$ is a product of at most $n+1$ mirrors.*

Proof. Let F be an isometry of $\mathbb{R}^n$. By repeatedly applying the above lemma, we may choose a sequence of hyperplanes $H_0, H_1, \ldots$ so that

$$\mathrm{Fix}\big(M_{H_{k-1}} \ldots M_{H_1} M_{H_0} F\big) \subseteq \mathrm{Fix}\big(M_{H_k} \ldots M_{H_1} M_{H_0} F\big)$$

The set of fixed points of each of these isometries is always an affine subspace, and these spaces are increasing in dimension. Our process must stop when we have reached a function with every point as a fixed point—that is when our function is equal to the identity—since at this point the lemma no longer applies. So $M_{H_0} F$ will have fixed points an affine space with dimension at least 0, $M_{H_1} M_{H_0} F$ will be an affine space with dimension and least 1, and so on. We deduce that there exists a nonnegative integer $k \leq n$ so that $M_{H_k} \ldots M_{H_1} M_{H_0} F = I$. Since each mirror is its own inverse, by left-multiplying the above equation by our mirrors we deduce that $F = M_0 M_1 \ldots M_{H_k}$ and this completes the proof. $\square$

Two Mirrors

Next we will look to see what transformation we see when we multiply two mirrors in the plane. Before this investigation, we will prove an extremely useful lemma for the purposes of identifying isometries.

Lemma 21.4 (Three points)**.** *Let F and G be isometries of the plane and let $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ be points in the plane that do not lie on a common line. If $F(\mathbf{x_i}) = G(\mathbf{x_i})$ holds for $i = 1, 2, 3$ then $F = G$.*

Proof. Let $\mathbf{y_i} = F(\mathbf{x_i}) = G(\mathbf{x_i})$ for $i = 1, 2, 3$. Now the isometry $G^{-1}F$ satisfies

$$G^{-1}F(\mathbf{x_i}) = G^{-1}(\mathbf{y_i}) = \mathbf{x_i} \quad \text{for } i = 1, 2, 3.$$

Since the points $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ do not lie on a line, it follows that $G^{-1}F = I$ so $F = G$. □

Directed Distance: If L and L' are parallel lines in the plane, the *directed distance* from L to L' is the vector $\mathbf{v} \in \mathbb{R}^2$ so that $\mathbf{v}$ is perpendicular to L and L' and translation by $\mathbf{v}$ maps L to L', that is $T_{\mathbf{v}}(L) = L'$.

Directed Angle: If L and L' are lines which meet at a unique point $\mathbf{x}$, the *directed angle* from L to L' is the counterclockwise angle from L to L' at $\mathbf{x}$.

Theorem 21.5. *Let L, L' be lines in $\mathbb{R}^2$. Then exactly one of the following holds:*

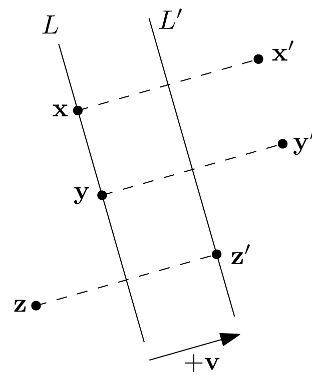
(i) *$L = L'$ and $M_{L'}M_L = I$.*

(ii) *L and L' are distinct and parallel and $M_{L'}M_L = T_{2\mathbf{v}}$ where $\mathbf{v}$ is the directed distance from L to L'.*

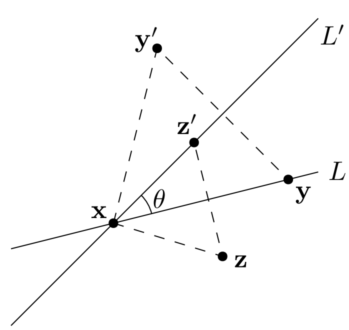
(iii) *L and L' meet at a unique point $\mathbf{x}$ and $M_{L'}M_L = R_{\mathbf{x},2\theta}$ where θ is the directed angle from L to L'.*

Proof. If $L = L'$ then we have $M_{L'}M_L = I$ as desired.
Next let L and L' be distinct parallel lines and let $\mathbf{v}$ be the directed distance from L to L'. Choose a point $\mathbf{x}$ on L. Then $M_L(\mathbf{x}) = \mathbf{x}$ so $M_{L'}M_L(\mathbf{x})$ is the point $\mathbf{x} + 2\mathbf{v}$. In other words, $M_{L'}M_L(\mathbf{x}) = T_{2\mathbf{v}}(\mathbf{x})$. Similarly, if $\mathbf{y}$ is another point on L distinct from $\mathbf{x}$ we must have $M_{L'}M_L(\mathbf{y}) = T_{2\mathbf{v}}(\mathbf{y})$. Next, choose a point $\mathbf{z'}$ on L' and set $\mathbf{z} = M_L(\mathbf{z'})$. Then we have

$\mathbf{z} = \mathbf{z}' - 2\mathbf{v}$. Furthermore we find that $M_{L'}M_L(\mathbf{z}) = M_{L'}(\mathbf{z}') = \mathbf{z}' = \mathbf{z} + 2\mathbf{v}$. Thus, again we have that $M_{L'}M_L(\mathbf{z}) = T_{2\mathbf{v}}(\mathbf{z})$. So, the transformations $M_{L'}M_L$ and $T_{2\mathbf{v}}$ send $\mathbf{x}, \mathbf{y}, \mathbf{z}$ to the same places, and it follows from Corollary 9.2 that $M_{L'}M_L = T_{2\mathbf{v}}$ as claimed.



Finally, we consider the case that L and L' intersect at a unique point $\mathbf{x}$ and we let θ be the directed angle from L to L'. It is immediate that $M_{L'}M_L(\mathbf{x}) = M_{L'}(\mathbf{x}) = \mathbf{x}$ so we have $M_{L'}M_L(\mathbf{x}) = R_{\mathbf{x},2\theta}(\mathbf{x})$. Next, choose a point $\mathbf{y}$ distinct from $\mathbf{x}$ on the line L. Then we find that $M_{L'}M_L(\mathbf{y}) = M_{L'}(\mathbf{y}) = \mathbf{y}'$ where L' is the perpendicular bisector of $\overline{\mathbf{y}\mathbf{y}'}$. But in this case it follows from the figure that $\mathbf{y}' = R_{\mathbf{x},2\theta}(\mathbf{y})$. Next, choose a point $\mathbf{z}'$ distinct from $\mathbf{x}$ on the line L' and let $\mathbf{z}' = M_L(\mathbf{z})$. Now we find that $M_{L'}M_L(\mathbf{z}) = M_{L'}(\mathbf{z}') = \mathbf{z}'$. But then (see figure) we have that $\mathbf{z}' = R_{\mathbf{x},2\theta}(\mathbf{z})$. Thus, the transformations $M_{L'}M_L$ and $R_{\mathbf{x},2\theta}$ send $\mathbf{x}, \mathbf{y}, \mathbf{z}$ to the same places, and by Corollary 9.2 we must have $M_{L'}M_L = R_{\mathbf{x},2\theta}$ as claimed. $\square$



Parity

Lemma 21.6. *If A is a orthogonal matrix, then*

- $A^{-1} = A^{\top}$
- $\det A = \pm 1$

Proof. Express A in terms of its column vectors as $A = \begin{bmatrix} \mathbf{a_1} & \ldots & \mathbf{a_n} \end{bmatrix}$ and note that $\mathbf{a_1}, \ldots, \mathbf{a_n}$ form an orthonormal basis. Now, $A^{\top}A$ is a matrix with (i,j) entry equal to the dot product of $\mathbf{a_i}$ and $\mathbf{a_j}$, so $A^{\top}A$ is the identity and we have $A^{-1} = A^{\top}$. The second part follows from $1 = \det I = \det A^{\top} \det A = (\det A)^2$ □

Definition. An isometry $F \in AGO_n$ given by $F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ is $\left\{ \begin{array}{c} \textit{even} \\ \textit{odd} \end{array} \right\}$ if $\left\{ \begin{array}{c} \det A = 1 \\ \det A = -1 \end{array} \right\}$.

Observation 21.7. *If $F, G \in AGO_n$ then the product FG satisfies*

$$FG \text{ is } \left\{ \begin{array}{c} \textit{even} \\ \textit{odd} \end{array} \right\} \text{ if } \left\{ \begin{array}{c} F, G \textit{ are either both even or both odd} \\ \textit{one of } F, G \textit{ is even and the other is odd} \end{array} \right\}.$$

Proof. Let F, G be given by $F(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$ and $G(\mathbf{x}) = B\mathbf{x} + \mathbf{d}$. Then FG is given by

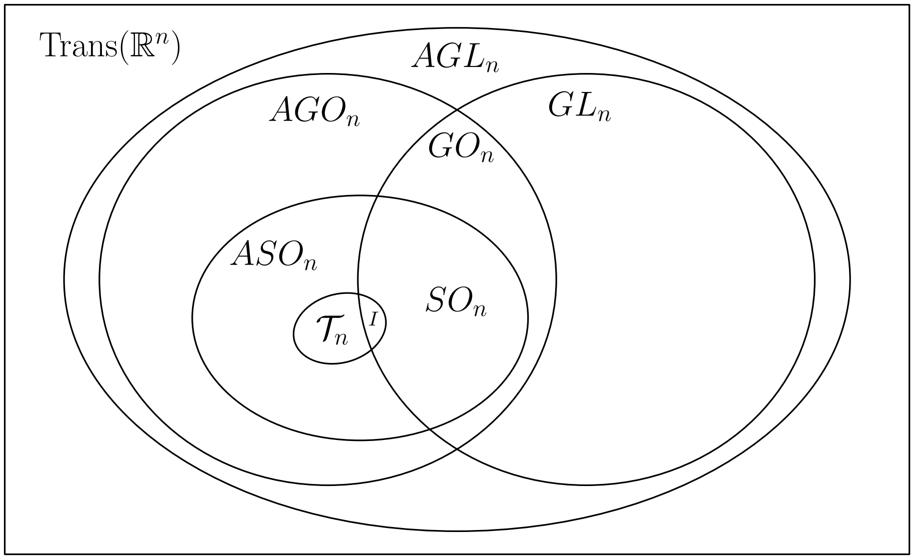
$$FG(\mathbf{x}) = F(G(\mathbf{x})) = F(B\mathbf{x} + \mathbf{d}) = AB\mathbf{x} + (A\mathbf{d} + \mathbf{c}).$$

Since $\det AB = \det A \det B$ and the result follows. □

Definition. We let SO_n be the subset of GO_n consisting of even functions and we let ASO_n be the subset of AGO_n consisting of even functions.

Lemma 21.8. *SO_n is a subgroup of GO_n and ASO_n is a subgroup of AGO_n*

Proof. The identity matrix has $\det I = 1$ so the identity function is always even. It follows that both SO_n and ASO_n contain the identity. It follows from the previous observation that both SO_n and ASO_n are closed under products. Finally, if F is an even function in SO_n or ASO_n then it follows from $AA^{-1} = I$ and the previous observation that A^{-1} is also even. □



Observation 21.9. *The parities of the plane isometries are:*

- *The identity, every translation, and every rotation are even.*
- *Every mirror and every glide is odd.*

Proof. In 2-dimensions, every orthogonal matrix has the form $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $M = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$. Here the matrix R gives a rotation about $\mathbf{0}$ and $\det R = 1$ while the matrix M gives a mirror about a line through $\mathbf{0}$ and $\det M = -1$. The observation follows from this and our analysis of the plane isometries. $\square$

Note: Now, just as we saw before with permutations and transpositions, we have that every isometry F of the plane can be expressed as a product of mirrors, and the number of mirrors determines the parity of F.