12 Extremal Graph Theory II

In this section, graphs are assumed to have no loops or parallel edges.

**Average Degree:** The average degree of a graph $G$ is
\[
\sum_{v \in V(G)} \deg(v) = \frac{2|E(G)|}{|V(G)|}.
\]

**Observation 12.1** For every $r \in \mathbb{N}$, every graph of average degree $\geq 2r$ contains a subgraph of minimum degree $\geq r + 1$.

**Proof:** We prove the observation by induction on $|V(G)|$. If $G$ has minimum degree $\geq r + 1$, then we are done. Otherwise, let $v \in V(G)$ satisfy $\deg(v) \leq r$. Then we have
\[
2\frac{|E(G - v)| + 2|\delta(v)|}{|V(G - v)| + 1} = \frac{2|E(G)|}{|V(G)|} \geq 2r \geq \frac{2|\delta(v)|}{1}.
\]
It follows from this (and the observation $a/b \geq c/d \Rightarrow a/c \geq a+b/c+d$) that $2\frac{|E(G - v)|}{|V(G - v)|} = \frac{2|E(G)|}{|V(G)|} \geq 2r$. (So, deleting a vertex of degree $\leq r$ from a graph with average degree $\geq 2r$ can only increase the average degree). Now, by induction, $G - v$ has a subgraph of minimum degree $\geq r + 1$ and this completes the proof. □

**Theorem 12.2 (Mader)** For every positive integer $r$, every graph with average degree $\geq 2\binom{r}{2}$ contains a subdivision of $K_r$.

**Proof:** We will prove by induction that for $m = r - 1, r, r + 1, \ldots, \binom{r}{2}$, every graph of average degree $\geq 2^m$ contains a subgraph which is a subdivision of a (simple) graph on $r$ vertices with $m$ edges. As a base, when $m = r - 1$, we may choose a vertex $v$ with $\deg(v) \geq 2^{r-1} \geq r - 1$. Now $v$ together with $r - 1$ of its neighbors induce a graph with $r$ vertices and $r - 1$ edges.

For the inductive step, we let $r \leq m \leq \binom{r}{2}$ and let $G$ be a graph with average degree $\geq 2^m$. We may assume (without loss) that $G$ is connected. Choose a maximal set $U \subseteq V(G)$ so that the graph induced on $U$ is connected and so that the graph $G'$ obtained from $G$ by identifying $U$ to a new vertex $u$ and then deleting all loops and parallel edges has average degree $\geq 2^m$. Note that such a set $U$ exists since $U = \{v\}$ works for every $v \in V(G)$.

Now, let $H$ be the subgraph of $G'$ induced by the neighbors of $u$. If there exists a vertex $v \in V(H)$ with $\deg_H(v) < 2^{m-1}$, then replacing $U$ by $U \cup \{v\}$ reduces the number of vertices in $G'$ by 1 and the number of edges by $< 2^{m-1}$ so this gives a $G''$ with average degree $\geq 2^m$, contradicting our choice of $U$. It follows that $H$ has average degree $\geq 2^{m-1}$. By induction,
we may choose a subgraph $K$ of $H$ which is a subdivision of an $r$ vertex graph with $m - 1$ edges. Since every point in $V(H)$ is adjacent to a point in $U$ and the graph induced on $U$ is connected, we may extend $K$ to a subgraph $K'$ of $G$ which is a subdivision of an $r$ vertex graph with $m$ edges, as desired. □

**Ball:** If $x \in V(G)$ and $n \in \mathbb{N}$, the ball of radius $n$ around $x$ is

$$B_n(x) = \{ v \in V(G) : \text{dist}(x, v) \leq n \}.$$  

**Theorem 12.3** Every graph $G$ with $\delta(G) \geq 3$ and no cycle of length $\leq 8\binom{r}{2} + 2$ contains $K_r$ as a minor.

**Proof:** Set $k = \binom{r}{2}$ and choose a maximal subset of vertices $X \subseteq V(G)$ with the property that $\text{dist}(x, y) \geq 2k + 1$ for every distinct $x, y \in X$. Now let $X = \{ x_1, \ldots, x_m \}$ and define a function $f : V(G) \to X$ by the rule that for every $v \in V(G)$, the vertex $f(v) = x_i$ is a point in $X$ with minimum distance to $v$, and subject to this has $i$ as small as possible. Now for $1 \leq i \leq m$ let $V_i = \{ v \in V(G) : f(v) = x_i \}$. The following inclusion follows immediately from our definitions (it holds for every $1 \leq i \leq m$).

$$B_k(x_i) \subseteq V_i \subseteq B_{2k}(x_i)$$

**Claim:** If $v \in V_i$ and $P$ is a shortest path from $v$ to $x_i$ then $V(P) \subseteq V_i$.

**Proof of Claim:** Suppose (for a contradiction) that $u \in V(P)$ satisfies $u \in V_j$ for $j \neq i$. If $\text{dist}(u, x_j) < \text{dist}(u, x_i)$, then we find $\text{dist}(v, x_j) < \text{dist}(u, x_i)$, which is contradictory. It follows from this (and $u \in V_j$) that $\text{dist}(u, x_j) = \text{dist}(u, x_i)$. From this we deduce $\text{dist}(v, x_j) = \text{dist}(v, x_i)$. However, now $v \in V_i$ implies $i < j$ and $u \in V_j$ implies $j < i$ which is a contradiction.

With this claim, we now deduce the following properties of $V_i$.

- The graph induced by $V_i$ is connected.
- $\text{dist}(v, x_i) \leq 2k$ for every $v \in V_i$.
- If $x, y \in V_i$ then $\text{dist}(x, y) \leq 4k$. 


For every $1 \leq i \leq m$ let $T_i$ be the graph induced on $V_i$. Suppose (for a contradiction) that $T_i$ has a cycle and choose such a cycle $C \subseteq T_i$ of minimum length. By assumption, $|E(C)| \geq 8k+2$, but then we may choose two points $u, v \in V(C)$ which are distance $\geq 4k+1$ on this cycle. Now there is a path $P \subseteq T_i$ from $u$ to $v$ of length $\leq 4k$ and now $P \cup C$ contains a shorter cycle than $C$, giving us a contradiction. Thus, we find that $T_i$ is a tree.

Now by degree sum arguments, we have $|\delta(V_i)| = \sum_{v \in V_i} \deg(v) - 2|E(T_i)| \geq 3|V_i| - 2|V_i| = |V_i| \geq |B_k(x_i)| \geq 2^k$

Now, construct a graph $G'$ from $G$ by contracting every edge in $T_i$ for every $1 \leq i \leq m$. It follows from the assumption that any two points in $T_i$ are joined by a path of length $\leq 4k$ in $T_i$ and the assumption that all cycles in $G$ have length $\geq 4k+3$ that $G'$ has no loops or parallel edges. So, $G'$ is a minor of $G$ which is simple with minimum degree $\geq 2^k = 2^{\binom{k}{2}}$, and by Theorem 12.2 we find that $G'$ contains a subdivision of $K_r$. This gives us a $K_r$ minor in $G$, as required. □

**Linking:** A graph $G$ is $k$-linked if $|V(G)| \geq 2k$ and for every $\{s_1, \ldots, s_k, t_1, \ldots, t_k\} \subseteq V(G)$, there exist vertex disjoint paths $P_1, \ldots, P_k$ so that $P_i$ has ends $s_i$ and $t_i$ for $1 \leq i \leq k$.

**Theorem 12.4** For every positive integer $k$, every $2^{\binom{3k}{2}}$-connected graph is $k$-linked.

**Proof:** Let $\{s_1, \ldots, s_k, t_1, \ldots, t_k\} \subseteq V(G)$ and apply Theorem 12.2 to choose a subdivision of $K_{3k}$ in $G$. Call this subdivision $H$ and let $U \subseteq V(G)$ be the set of vertices with degree $3k$ in $H$. By Menger’s Theorem, we may choose a collection of paths $Q_1, \ldots, Q_{2k} \subseteq G$ so that each of these paths has a distinct starting point in $\{s_1, \ldots, s_k, t_1, \ldots, t_k\}$ and a distinct endpoint in $U$. Subject to this, choose $Q_1, \ldots, Q_{2k}$ so that each $Q_i$ has a minimum number of edges in $E(G) \setminus E(H)$.

Let $U = \{s'_1, \ldots, s'_k, t'_1, \ldots, t'_k, u_1, \ldots, u_k\}$ and assume that for $1 \leq i \leq k$ the path $Q_i$ has ends $s_i$ and $s'_i$ and that the path $Q_{k+i}$ has ends $t_i$ and $t'_i$. Let $1 \leq i \leq k$ and let $R_i, R'_i \subseteq H$
be the paths from \( u_i \) to \( s'_i \) and \( t'_i \) which correspond to (possibly subdivided) edges of our \( K_{3k} \).

Let \( v_i \) be the first vertex on the path \( R_i \) from \( u_i \) to \( s'_i \) which is contained in one of the paths \( Q_1, \ldots, Q_{2k} \) and suppose that \( v_i \in Q_j \). Now, consider rerouting \( Q_j \) along \( R_i \) to the vertex \( u_i \). The resulting paths \( Q_1, \ldots, Q_{2k} \) would be vertex disjoint, so it follows from our choice that \( Q_j \) uses no edges in \( E(G) \setminus E(H) \) after \( v_i \). It follows from this that \( j = i \) and after the vertex \( v_i \), the path \( Q_i \) follows \( R_i \) to \( s'_i \). By a similar argument, we find that the first vertex \( v'_i \) on \( R'_i \) which lies on one of \( Q_1, \ldots, Q_{2k} \) is on the path \( Q_{k+i} \). Now, define the path \( P_i \) to be the path from \( s_i \) to \( t_i \) obtained by following \( Q_i \) from \( s_i \) to \( v_i \), then following \( R_i \) to \( u_i \), then following \( R'_i \) to \( v'_i \) and then following \( Q_{k+i} \) to \( t_i \). It is a consequence of our construction that \( P_1, \ldots, P_k \) are vertex disjoint, thus completing the proof. \( \square \)