

Mathematical Induction

Induction is an incredibly powerful tool for proving theorems in discrete mathematics. In this document we will establish the proper framework for proving theorems by induction, and (hopefully) dispel a common misconception.

Basic induction

Basic induction is the simplest to understand and explain. Suppose you wish to prove that for every positive integer n the property $P(n)$ holds. Then, instead of showing this all at once, it suffices to prove the following two properties.

- (i) $P(1)$ holds
- (ii) If $P(n - 1)$ holds, then $P(n)$ holds.

We call (i) the *base case* and (ii) the *inductive step*. You can think of this like dominos. We have a big string of dominoes, numbered $1, 2, 3, \dots$. If you knock the first one over (the base case), and each one knocks over the next (the inductive step), then every domino will be knocked down. Here is an example.

Proposition 1 $\sum_{i=1}^n (2i - 1) = n^2$ for every positive integer n .

Proof: We proceed by induction on n . As a base case, observe that when $n = 1$ we have $\sum_{i=1}^1 (2i - 1) = 1 = 1^2$. For the inductive step, let $n > 1$ be an integer, and assume that the proposition holds for $n - 1$. Now we have

$$\begin{aligned} \sum_{i=1}^n (2i - 1) &= \sum_{i=1}^{n-1} (2i - 1) + 2n - 1 \\ &= (n - 1)^2 + 2n - 1 \\ &= n^2. \end{aligned}$$

Thus, the proposition holds for n , and this completes the proof. \square .

Strong induction

Strong induction is (not surprisingly) a strengthening of the basic type. This is the usual framework we use when we try to prove statements in discrete math by induction. Here again, you wish to prove that for every positive integer n , the property $P(n)$ holds. The base case is the same as before, but now you utilize a stronger inductive assumption. Namely, to employ this method you must show the following.

- (i) $P(1)$ holds
- (ii) If $P(k)$ holds for all $k < n$, then $P(n)$ holds.

As before, we call (i) the *base case* and (ii) the *inductive step*. In terms of our dominos, you show that the first one is knocked over (the base case), and then you show that if the dominos $1 \dots n - 1$ have fallen, then domino n falls as well (the inductive step). If both the base case and the inductive step have been shown, then all of the dominos will fall.

This technique is only slightly different from basic induction, and we don't waste any space in our proofs distinguishing between them. However, you should appreciate that this form of induction gives you more information (=power). When you go to show that $P(n)$ is true, you are equipped with the fact that $P(k)$ is true for all $k < n$, not just for $k = n - 1$. Next we exhibit an example of an inductive proof in graph theory.

Theorem 2 *Every connected graph G with $|V(G)| \geq 2$ has at least two vertices x_1, x_2 so that $G - x_i$ is connected for $i = 1, 2$.*

Proof: We proceed by induction on $|V(G)|$. As a base case, observe that if G is a connected graph with $|V(G)| = 2$, then both vertices of G satisfy the required conclusion. For the inductive step, let G be a connected graph with $|V(G)| \geq 2$ and assume that the theorem holds for every graph with $< |V(G)|$ vertices. If $G - x$ is connected for every vertex $x \in V(G)$, then we are done, so we may assume this is not so, and choose $x \in V(G)$ so that $G - x$ has components H_1, H_2, \dots, H_m where $m \geq 2$. For every $1 \leq i \leq m$ let H'_i be the graph obtained from H_i by adding back the vertex x and all edges with one end x and the other end in $V(H_i)$. So every H'_i is a connected graph with at least two vertices. Furthermore, $|V(H'_i)| < |V(G)|$, so by induction, H'_i must have at least one vertex $x_i \neq x$ so that $H'_i - x_i$ is connected. It then follows that $G - x_i$ is connected. Since we have such an x_i for every component (and at least two components), this completes the proof. \square

Let's pause here to make a few observations about this proof. First, notice that we never formally defined our expression $P()$ - indeed, we never even gave a name to the inductive parameter $|V(G)|$. Of course, this would not be difficult to do if we wanted: for every $n \geq 2$ we define $P(n)$ to be the property that the theorem holds for all graphs on n vertices. Then, in the proof, we have checked the base case $P(2)$, and then for the inductive step, we showed that $P(n)$ holds, assuming $P(k)$ for all $k < n$. If you don't need them, you can save clutter by not introducing n and k .

Notice that when we wish to prove the inductive step, we let G be an *arbitrary* graph which satisfies the assumptions, and we assume that the statement is true for all graphs with fewer vertices. This is vital. It is a common misconception that the inductive step for the n^{th} level may be established by beginning with an example of level $n - 1$ and then "building" something on the n^{th} level out of this. This is logically incorrect. You must take an *arbitrary* object at the n^{th} level, and then prove this satisfies the theorem using the assumption that the theorem holds for all objects at lower levels. In hopes of clarifying the common mistake, we exhibit here a false theorem.

False Theorem 3 *Every graph G with $|V(G)| \geq 3$ and all vertices of degree ≥ 2 contains a cycle of length 3.*

False Proof: We proceed by induction on $|V(G)|$. As a base case, observe that the theorem is true when $|V(G)| = 3$, since any simple graph on three vertices with all vertices of degree ≥ 2 must be a cycle of length 3. To prove the inductive step, let G be a graph on $n - 1$ vertices for which the theorem holds, and construct a new graph G' on n vertices by adding one new vertex to G and ≥ 2 edges incident with this new vertex. Since G contained a cycle of length 3, the graph G' also contains a cycle of length 3. This completes the proof. \square

Hopefully everyone can see that this theorem is incorrect - for instance it fails when you take G to be a cycle of length ≥ 4 . Moreover, it should be apparent what the mistake is: It is illegal to construct your graph on n vertices by starting with one on $n - 1$ where the result holds and then adding a vertex. This procedure is not allowed, since it does not produce all possible graphs on n vertices which satisfy the assumptions (all vertices of degree ≥ 2). To avoid this pitfall, when proving the inductive step, you should *always* let your object be an *arbitrary* object at the n^{th} level, and then prove that this object satisfies the statement assuming that it is true for all objects at (strictly) lower levels.