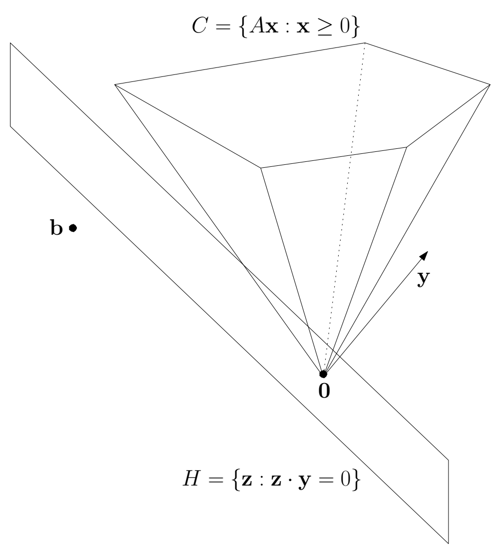
Linear Programming

Vectors in this section are column vectors by default. The dimensions of our vectors are frequently not stated, but must be inferred from context. If a, b are vectors from the same space, we write $a \leq b$ if $a_i \leq b_i$ for every coordinate i. Similarly, we write $a \geq 0$ if a is coordinatewise greater than the vector of zeros.

Cone: A set $C \subseteq \mathbb{R}^n$ is a *cone* if $\lambda x \in C$ whenever $x \in C$ and $\lambda \geq 0$.

Polyhedral Cone: A *polyhedral cone* is any set of the form $\{Ax : x \geq 0\}$ where A is a real $m \times n$ matrix.



Lemma 1 (Farkas Lemma) *If A is an $m \times n$ real matrix and $b \in \mathbb{R}^m$, then exactly one of the following holds:*

(i) *There exists $x \geq 0$ so that $Ax = b$.*

(ii) *There exists y so that $y^\top A \geq 0$ and $y^\top b < 0$.*

Note: Lemma 1 is equivalent to the obvious fact that given a point b and a cone $C = \{Ax : x \geq 0\}$, either (i) $b \in C$ or (ii) there is a hyperplane (with normal y) through the origin separating b from C.

Proof: It follows from the fact that C is closed and convex that there exists a hyperplane H with normal vector y which separates b from C. Shift H to a parallel hyperplane H' (keeping the same normal vector) until it meets the cone C. Since 0 is in every minimal face of C, it follows that $0 \in H'$. Now by possibly replacing y by $-y$ we may arrange that $y^\top b < 0$ and $y^\top A \geq 0$.

Corollary 2 *If A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, then exactly one of the following holds:*

(i) *There exists x so that $Ax \leq b$.*

(ii) *There exists $y \geq 0$ so that $y^\top A = 0$ and $y^\top b < 0$.*

Proof: It is immediate that (i) and (ii) are mutually exclusive, as otherwise we would have $0 = y^\top Ax \leq y^\top b < 0$ which is contradictory.

To see that one of these conclusions must hold, consider the matrix $A' = [I \ A \ -A]$ and apply the Farkas Lemma to A' and b. If there exists a vector $z^\top = [w^\top, x_p^\top, x_m^\top] \geq 0$ so that $A'z = b$, then we have that $A(x_p - x_m) \leq b$, so (i) holds. Otherwise, there must be a vector y so that $y^\top A' \geq 0$ and $y^\top b < 0$, but then $y \geq 0$ and $y^\top A = 0$ so (ii) holds. □

Linear Programming: Fix an $m \times n$ matrix A and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. A *linear program* and the associated *dual* are given as follows:

LP (primal)	Dual
maximize $c^\top x$	minimize $y^\top b$
s.t. $Ax \leq b$	s.t. $y^\top A = c$, $y \geq 0$

We say that a point x (y) satisfying $Ax \leq b$ ($y^\top A = c$ and $y \geq 0$) is a *feasible* point for the linear program (dual). If no such point exists the problem is called *infeasible*. If the primal (dual) problem is feasible but has no maximum (minimum), it is called *unbounded*.

Observation 3 (Weak Duality) *If x is feasible for the Linear Program and y is feasible for the dual, then*

$$c^\top x \leq y^\top b$$

Proof: $c^\top x = y^\top Ax \leq y^\top b$ □

Note: It follows from the above that any feasible point in the dual gives an upper bound on the primal problem (and vice versa). So, in particular, if the dual problem is feasible, then the primal problem is bounded.

Theorem 4 (Strong Duality) *If the primal and dual problem are feasible, then the optimum points x, y satisfy $c^\top x = y^\top b$.*

Proof: Consider the following equation

$$\begin{bmatrix} A & 0 \\ -c^\top & b^\top \\ 0 & A^\top \\ 0 & -A^\top \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} b \\ 0 \\ c \\ -c \\ 0 \end{bmatrix}$$

If there exist x, y satisfying the above equation, then $Ax \leq b$ so x is feasible, $y \geq 0$ and $y^\top A = c^\top$ so y is feasible. Further $-c^\top x + b^\top y \leq 0$ so $y^\top b \leq c^\top x$ and by Weak Duality we must then have $y^\top b = c^\top x$ and we are finished. Otherwise, by Corollary 2 there exists $[y^\top, \lambda, x_m^\top, x_p^\top, w^\top] \geq 0$ satisfying

$$[y^\top, \lambda, x_m^\top, x_p^\top, w^\top] \begin{bmatrix} A & 0 \\ -c^\top & b^\top \\ 0 & A^\top \\ 0 & -A^\top \\ 0 & -I \end{bmatrix} = 0 \qquad \text{and} \qquad [y^\top, \lambda, x_m^\top, x_p^\top, w^\top] \begin{bmatrix} b \\ 0 \\ c \\ -c \\ 0 \end{bmatrix} < 0$$

This gives the following:

$$y^\top A = \lambda c^\top \quad \text{and} \quad y \geq 0 \tag{1}$$

$$A(x_p - x_m) \leq \lambda b \tag{2}$$

$$y^\top b < c^\top (x_p - x_m) \tag{3}$$

If $\lambda > 0$, then scaling the vector $[y^\top, \lambda, x_m^\top, x_p^\top, w^\top]$ by $1/\lambda$ we may assume that $\lambda = 1$. However, then (1) and (2) show that $x_p - x_m$ and y are feasible in the primal and dual (respectively) and (3) contradicts Weak Duality.

Otherwise we have $\lambda = 0$. Now, by (3), either $y^\top b < 0$ or $c^\top (x_p - x_m) > 0$. In the former case, we claim that the dual problem is unbounded (which contradicts the assumption that the primal is feasible). To see this, let y_f be any feasible point in the dual, let μ be a positive number, and consider the vector $y_f + \mu y$. We have $y_f + \mu y \geq 0$ and $(y_f + \mu y)^\top A =$

$y_f^\top A + \mu y^\top A = b^\top$, so this vector is feasible in the dual, and $(y_f + \mu y)^\top b = y_f^\top b + \mu(y^\top b)$ can be made arbitrarily small by choosing μ sufficiently large. If $c^\top(x_p - x_m) > 0$ then a similar argument shows that the primal is unbounded (again giving us a contradiction). $\square$