Arkhipov's parity theorem

Motivated by some questions in quantum computing related to the Kochen-Specker theorem, Alex Arkhipov proved a lovely result in his Masters thesis which characterizes planar graphs. In this note we give a straightforward generalization of one direction of Arkhipov's theorem. This gives a kind of global "parity" condition for a type of flow-like function on an embedded planar graph.

For a digraph $D = (V, E)$, we view each edge $e \in E$ with tail u and head v as composed of two half-edges, one of which is incident to u and the other incident to v. So each half-edge h is incident to exactly one vertex, which we denote by v_h, and is contained in exactly one edge, which we denote by e_h. The *sign* of a half-edge h is defined to be

$$\sigma(h) = \begin{cases} 1 & \text{if } v_h \text{ is the tail of } e_h \\ -1 & \text{if } v_h \text{ is the head of } e_h \end{cases}$$

If D is embedded in an orientable surface, then each vertex v is endowed with a clockwise ordering of the incident half-edges. We describe this using a "rotation scheme" π: We let $\pi(v) = h_1, \ldots, h_k$ if the clockwise cyclic ordering of half-edges incident with v may be expressed as $h_1, \ldots, h_k$ and then back to h_1. Naturally, we treat this rotation scheme as equivalent to that where $\pi(v)$ is replaced by $h_2, \ldots, h_k, h_1$ or any other cyclic shift.

Theorem 1 (Arkhipov) *Let $D = (V, E)$ be a digraph embedded in the plane, let G be a group with centre Z, and let $\phi : E \to G$. Assume that every $v \in V$ with $\pi(v) = h_1, \ldots, h_k$ satisfies*

$$\phi(v) = \prod_{i=1}^{k} \phi\left(e_{h_i}\right)^{\sigma(h_i)} \in Z.$$

Note that in this case the product given by $\phi(v)$ is invariant under modifying $\pi(v)$ by a cyclic shift (as the new product is a conjugate of the original). In this case

$$\prod_{v \in V} \phi(v) = 1.$$

Here the order in which this product is computed does not matter as the elements all lie in Z.

Proof: We proceed by induction on $|E|$. As a base case, note that the result holds trivially when $|E| = 0$. For the inductive step, first suppose there exists an edge e with distinct endpoints u, v. Assume that $\pi(u) = h_1, \ldots, h_k$ and $\pi(v) = h'_1, \ldots, h'_\ell$ where $e_{h_k} = e = e_{h'_1}$. Since $\sigma(h_k)\sigma(h'_1) = -1$, we have

$$\begin{aligned}\phi(u)\phi(v) &= \phi\left(e_{h_1}\right)^{\sigma(h_1)} \ldots \phi\left(e_{h_k}\right)^{\sigma(h_k)} \phi\left(e_{h'_1}\right)^{\sigma(h'_1)} \ldots \phi\left(e_{h'_\ell}\right)^{\sigma(h'_\ell)} \\ &= \phi\left(e_{h_1}\right)^{\sigma(h_1)} \ldots \phi\left(e_{h_{k-1}}\right)^{\sigma(h_{k-1})} \phi\left(e_{h'_2}\right)^{\sigma(h'_2)} \ldots \phi\left(e_{h'_\ell}\right)^{\sigma(h'_\ell)}\end{aligned}$$

Now consider the embedded digraph D/e obtained by contracting the edge e to form a new vertex w. The last term in the above equation is precisely $\phi(w)$ for this new vertex, so $\phi(w) \in Z$. By induction, the theorem is satisfied for D/e, and then (using the above equation) we find that the theorem also holds for D.

In the remaining case every edge is a loop, and by planarity, there must exist such an edge e incident with a vertex v so that the half-edges contained in e are consecutive at v. Assume that $\pi(v) = h_1, \ldots, h_k$ where $e_{h_1} = e = e_{h_2}$. Now $\sigma(h_1)\sigma(h_2) = -1$ so

$$\phi(v) = \phi\left(e_{h_1}\right)^{\sigma(h_1)} \ldots \phi\left(e_{h_k}\right)^{\sigma(h_k)} = \phi\left(e_{h_3}\right)^{\sigma(h_3)} \ldots \phi\left(e_{h_k}\right)^{\sigma(h_k)}$$

The result now follows from the above equation and induction on the digraph obtained from D by deleting the edge e. $\square$