Fisher's Theorem

Fix a simple digraph $D = (V, E)$, let $v \in V$, and let $k \in \mathbb{Z}$. If $k \geq 0$ we let $N^k_D(v)$ denote the set of vertices at distance k from v, and if $k < 0$ we let $N^k_D(v)$ denote the set of vertices with distance $-k$ to v. We define $deg^k_D(v) = |N^k_D(v)|$, and (as usual) we drop the subscripts from these when the graph is clear from context. The purpose of this note is to prove the following theorem which was originally conjectured by Dean.

Theorem 1 (Fisher) *Every tournament has a vertex v with $deg^2(v) \geq deg^1(v)$.*

We note that Seymour has conjectured that the above result holds more generally for every digraph without a digon (directed cycle of length two), but this remains open. Our proof of Fisher's theorem requires the following key tool from linear programming. Here we treat elements of $\mathbb{R}^k$ as column vectors.

Lemma 2 (Farkas) *If B is a real $m \times n$ matrix and $c \in \mathbb{R}^n$, exactly one of the following holds.*

(i) *There exists $x \in \mathbb{R}^m$ with $x \geq 0$ so that $Bx = c$.*

(ii) *There exists $y \in \mathbb{R}^n$ so that $y^\top B \geq 0$ and $y^\top c < 0$.*

We say that a probability distribution $p : V \to \mathbb{R}$ *losing* if $p(N^1(v)) \geq p(N^{-1}(v))$ for every vertex v.

Lemma 3 *If D has no digon, then it has a losing distribution.*

Proof: Let $V = \{v_1, v_2, \ldots, v_n\}$ and let $A = \{a_{i,j}\}_{i,j \in \{1,\ldots,n\}}$ be the matrix given by the rule

$$a_{i,j} = \begin{cases} -1 & \text{if } (v_i, v_j) \in E \\ 1 & \text{if } (v_j, v_i) \in E \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbf{1}$ ($\mathbf{0}$) denote the vector in $\mathbb{R}^n$ of 1's (0's). Note that a losing distribution is precisely a vector $p \in \mathbb{R}^n$ with $p \geq 0$, $p^\top \mathbf{1} = 1$ and $Ap \leq 0$. Now, consider the following equation with variables $p, s \in \mathbb{R}^n$

$$\begin{bmatrix} A & I \\ \mathbf{1}^\top & \mathbf{0}^\top \end{bmatrix} \begin{bmatrix} p \\ s \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$$

If there exists a solution to the equation with $p, s \geq 0$, then p is a losing distribution and we are done. Otherwise, it follows from Farkas' Lemma that there exists $q \in \mathbb{R}^n$ and $t \in \mathbb{R}$ so that

$$[q^\top t] \begin{bmatrix} A^\top & I \\ \mathbf{1}^\top & \mathbf{0}^\top \end{bmatrix} \geq 0 \qquad \text{and} \qquad [q^\top t] \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} < 0.$$

However, in this case we must have $q \geq 0$ and $t < 0$ and these imply further that $q^\top A \geq 0$. But then $0 \leq A^\top q = -Aq$ so q is a nonnegative vector with $Aq \leq 0$ and then $\frac{1}{\mathbf{1}^\top q} q$ is a losing distribution. This completes the proof. $\square$

Observation 4 *If $k \in \mathbb{Z}$, and we choose $u \in V$ according to the probability distribution $p : V \to \mathbb{R}$, then*

$$\mathbb{E}(deg^k(u)) = \sum_{w \in V} p(N^{-k}(w))$$

Proof: We shall assume $k \geq 0$, the other case is similar.

$$\begin{aligned} \mathbb{E}(deg^k(u)) &= \sum_{v \in V} p(v) deg^k(v) \\ &= \sum_{v,w \in V : dist(v,w)=k} p(v) \\ &= \sum_{w \in V} p(N^{-k}(w)). \qquad \square \end{aligned}$$

Proof of Fisher's Theorem: Let $D = (V, E)$ be a tournament. By Lemma 3, we may choose a losing distribution $p : V \to \mathbb{R}$. We shall show that if a vertex u is chosen according to this distribution, then the expected size of $deg^2(u)$ is at least the expected size of $deg^1(u)$. By the previous observation, to do this, it suffices to show the following.

Claim: $p(N^{-2}(u)) \geq p(N^{-1}(u))$ for every $u \in V$.

To prove the claim, let $u \in V$, let $R = \{u\} \cup N^{-1}(u) \cup N^{-2}(u)$, and let Q be the tournament $T - R$. If the total weight on the vertices in Q is zero, then we have $p(N^{-2}(u)) = p(N^1(u)) \geq p(N^{-1}(u))$. Thus, we may assume $p(V(Q)) \geq 0$. It follows immediately from

$$\sum_{w \in V(Q)} p(w)(p(N_Q^1(w)) - p(N_Q^{-1}(w))) = \sum_{(w,x) \in E(Q)} p(w)p(x) - \sum_{(y,w) \in E(Q)} p(w)p(y) = 0$$

that there exists a vertex $w \in V(Q)$ so that $p(N_Q^{-1}(w)) \geq p(N_Q^1(w))$. Now, w must satisfy $p(N^1(w)) \geq p(N^{-1})(w)$ since p is losing, but then we must have

$$p(N^1(w) \cap R) \geq p(N^{-1}(w) \cap R). \tag{1}$$

Since $w \notin N^{-2}(u) \cup N^{-1}(u)$ we have

$$N^1(w) \cap R \subseteq N^{-2}(u). \tag{2}$$

Now, combining (1) and (2) yields

$$p(N^{-2}(u)) \geq p(N^1(w) \cap R) \geq p(N^{-1}(w) \cap R) \geq p(N^{-1}(u))$$

which completes the proof. $\square$