The Gallai-Edmonds Decomposition

Here we present Kotlov's proof of the Gallai-Edmonds decomposition. For every graph G, we let $odd(G)$ denote the number of odd components of G (i.e., components $H \subseteq G$ with $|V(H)|$ odd) . The following famous theorem of Tutte gives a necessary and sufficient condition for the existence of a perfect matching.

Theorem 1 (Tutte) *A graph G has a perfect matching if and only if $odd(G - X) \leq |X|$ for every $X \subseteq V(G)$.*

This theorem has a sharper formulation which gives considerable information about the structure of maximum size matchings. For every $X \subseteq V(G)$ we let $N(X) = \{y \in V(G) \setminus X :$ there exists $x \in X$ with $xy \in E(G)\}$. We say that G is *hypomatchable* if $G - v$ has a perfect matching for every $v \in V(G)$.

Theorem 2 (Edmonds-Gallai) *Let G be a graph, and let $A \subseteq V(G)$ be the collection of all vertices v so that there exists a maximum size matching which does not cover v. Set $B = N(A)$ and $C = V(G) \setminus (A \cup B)$. Then:*

- *Every odd component H of $G - B$ is hypomatchable and has $V(H) \subseteq A$.*
- *Every even component H of $G - B$ has a perfect matching and has $V(H) \subseteq C$.*
- *For every $X \subseteq B$, the set $N(X)$ contains vertices in $> |X|$ odd components of $G - B$.*

Note that the last condition above implies $odd(G - B) > |B|$ whenever $B \neq \emptyset$

Proof: We proceed by induction on $|V(G)|$. Choose a set $B \subseteq V(G)$ so that

(i) $odd(G - B) - |B|$ is maximum.

(ii) the number of non-hypomatchable components of $G - B$ is minimum (subj. to (i)).

(iii) $|B|$ is minimum (subj. to (i) and (ii)).

We shall establish properties of B in steps.

(1) Every even component of $G - B$ has a perfect matching.

If H is an even component of $G - B$ which has no perfect matching, then by Tutte's theorem there exists $Y \subseteq V(H)$ so that $odd(H - Y) > |Y|$. But then $B \cup Y$ contradicts our choice of B for (i).

(2) Every odd component of $G - B$ is hypomatchable.

If H is an odd component of $G - B$ which is not hypomatchable, then we may choose $v \in V(H)$ so that $H - v$ has no perfect matching. Now, apply the theorem inductively to $H - v$ to obtain a set $Y \subseteq V(H - v)$ so that $odd((H - v) - Y) > |Y|$ and every odd component of $(H - v) - Y$ is hypomatchable. Then $B \cup Y \cup \{v\}$ contradicts our choice of B for (ii) (or (i)).

(3) For every $X \subseteq B$, the set $N(X)$ has vertices in $> |X|$ odd components of $G - B$.

If $N(X)$ intersects the vertex set of fewer than $|X|$ odd components of $G - B$, then $B \setminus X$ contradicts the choice of B for (i). If $X \subseteq B$ intersects the vertex set of exactly $|X|$ odd components of $G - B$, then $B \setminus X$ contradicts the choice of B for (iii).

It follows from (1), (2), and (3) that every matching M of maximum size must match each vertex in B with a vertex in a distinct odd component of $G - B$. Furthermore, setting A to be the union of the vertex sets of odd components of $G - B$, and C to be the union of the vertex sets of the even components of $G - B$, we have that A, B, C are exactly the sets described in the statement of the theorem. Thus, (1), (2), and (3) imply the result. $\square$