Colouring Kneser Graphs

Kneser Graph: Let n and k be positive integers with $n \geq 2k$ and define the *Kneser Graph* $Kn(n,k)$ as follows. The vertex set is the collection of all k-element subsets of $\{1,2,\ldots,n\}$ and two vertices are adjacent if they are disjoint.

The following result was famously conjectured by Kneser and proved by Lovász.

Theorem 1 (Lovász) $\chi(Kn(n,k)) = n - 2k + 2$.

Our proof of this theorem is due to Greene, and is based on the Borsuk-Ulam Theorem which we shall state but not prove.

Spheres: The n-sphere is $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$. We say that $x, y \in S^n$ are *antipodes* or *are antipodal* if $y = -x$.

Theorem 2 (Borsuk-Ulam) *If $f : S^n \to \mathbb{R}^n$ is continuous, then there exists $x \in S^n$ with $f(x) = f(-x)$.*

Theorem 3 (Lusternik-Shnirelman-Borsuk) *Let $U_1, \ldots, U_{n+1}$ be subsets of S^n each of which is either open or closed and assume that $\cup_{i=1}^{n+1} U_i = S^n$. Then there exists $1 \leq i \leq n+1$ so that U_i contains a pair of anitpodal points.*

Proof: First we consider the case that each U_i is closed. Define the function $f : S^n \to \mathbb{R}^n$ by the rule $f(x) = (dist(x, U_1), dist(x, U_2), \ldots dist(x, U_n))$. It is immediate that f is continuous, so by the Borsuk-Ulam Theorem there must exist $x, -x \in S^n$ with $f(x) = f(-x)$. If the vector $f(x) = f(-x)$ has coordinate i equal to zero then $x, -x \in U_i$ and we are done. Otherwise, neither x nor $-x$ is contained in U_i for $1 \leq i \leq n$ so we must have $x, -x \in U_{n+1}$.

Next consider the case that each U_i is open. In this case, we may choose for each point $x \in S^n$ a set U_i containing x and an open ball around x, say B_x, so that the closure of B_x is contained in U_i (we have invoked the Axiom of Choice here, but it can be avoided). Now the family $\{B_x\}_{x \in S^n}$ is an open cover of the compact set S^n so we may choose a finite subcover of it $\{B_x\}_{x \in I}$. Next, for each $1 \leq i \leq n+1$ define U_i' to be the union of all $\overline{B_x}$ with $x \in I$ for which $\overline{B_x} \subseteq U_i$. By construction, each U_i is closed and $U_1', \ldots, U_{n+1}'$ is a cover of S^n so by the previous case, there must exist $1 \leq i \leq n+1$ so that U_i' contains a pair of antipodes. However, $U_i' \subseteq U_i$ so U_i must contain an antipodal pair.

We now prove the general case by induction on the number of closed sets. We have already resolved the base case when every U_i is open, so it only remains to prove the inductive step, for which we may assume that U_i is closed. If U_i contains a pair of antipodes, then we are finished, so we may assume otherwise. In this case, the maximum distance between a pair of points in U_i is $2-\epsilon$ for some $\epsilon > 0$. Define U_i' to be the set of all points in S^n at distance $< \frac{\epsilon}{2}$ from U_i. It is immediate that U_i' is open and does not contain a pair of antipodes. Now applying the induction hypothesis with U_i' in place of U_i yields the desired result. □

Proof of Theorem 1: To colour $Kn(n,k)$ with $n-2k+2$ colours, assign each vertex v which contains an element in $\{1,\ldots,n-2k\}$ to the minimum element of v (it is immediate that these colour classes are independent sets). The remaining unassigned vertices are k element subsets of $\{n-2k+1,\ldots,n\}$ (a set of size $2k$) so they induce a perfect matching which can be coloured with the use of two added colours.

For the other direction, set $d = n-2k$ and suppose (for a contradiction) that we have a $(d+1)$-colouring of $Kn(n,k)$. Now, let $Q = \{1,2,\ldots,n\}$ and identify each point of Q with a point of S^{d+1} in such a way that the points in Q lie in general position (i.e. no $d+2$ of them lie on any great d-sphere). For any point $x \in S^{d+1}$ we let $H(x)$ denote the open hemisphere centered at x. Next, for each $1 \le i \le d+1$ define the set U_i to consist of all points $x \in S^{d+1}$ with the property that $H(x)$ contains a k-element subset of Q with colour i. It is immediate that each U_i is open so the set $F = S^{d+1} \setminus (\cup_{i=1}^{d+1} U_i)$ is closed. Together $U_1,\ldots,U_{d+1},F$ cover S^{d+1} so by the Theorem 3, one of these sets must contain an antipodal pair of points. If F contains an antipodal pair $x,-x$ then $H(x)$ and $H(-x)$ each contain at most $k-1$ points from Q. But then the great d-sphere between x and $-x$ contains at least $n-2k+2 = d+2$ points which contradicts our assumption of general position. Otherwise, some U_i contains a pair of antipodal points $x,-x$. In this case, both $H(x)$ and $H(-x)$ contain a k-element subset of Q with colour i and these subsets are disjoint. However, these are two adjacent vertices in $Kn(n,k)$ which have the same colour, giving us a contradiction. □