Kneser's Addition Theorem

Throughout we shall assume that G is an additive abelian group. If $A, B \subseteq G$ and $g \in G$, then $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$ and $A + g = g + A = \{a + g \mid a \in A\}$. We define the *stabilizer* of A to be $\mathcal{S}(A) = \{g \in G \mid A + g = A\}$. Note that $\mathcal{S}(A) \leq G$. Our goal here is to prove the following theorem.

Theorem 1 (Kneser) *If $A, B \subseteq G$ are finite and nonempty and $K = \mathcal{S}(A + B)$, then*

$$|A + B| \geq |A + K| + |B + K| - |K|.$$

Proof. We proceed by induction on $|A + B| + |A|$. Suppose that $K \neq \{0\}$ and let $\phi : G \to G/K$ be the canonical homomorphism. Then $\mathcal{S}(\phi(A + B))$ is trivial, so by applying induction to $\phi(A), \phi(B)$ we have

$$|A + B| = |K|(|\phi(A) + \phi(B)|) \geq |K|(|\phi(A)| + |\phi(B)| - 1) = |A + K| + |B + K| - |K|.$$

Thus, we may assume $K = \{0\}$. If $|A| = 1$, then the result is trivial, so we may assume $|A| > 1$ and choose distinct $a, a' \in A$. Since $a' - a \notin \mathcal{S}(A+B) \supseteq \mathcal{S}(B)$, we may choose $b \in B$ so that $b + a' - a \notin B$. Now by replacing B by $B - b + a$ we may assume $\emptyset \neq A \cap B \neq A$.

Let $C \subseteq A + B$ and let $H = \mathcal{S}(C)$. We call C a *convergent* if

$$|C| + |H| \geq |A \cap B| + |(A \cup B) + H|.$$

Set $C_0 = (A \cap B) + (A \cup B)$ and observe that $C_0 \subseteq A + B$. Since $0 < |A \cap B| < |A|$, we may apply induction to $A \cap B$ and $A \cup B$ to conclude that C_0 is a convergent. Thus a convergent exists, and we may now choose a convergent C with $H = \mathcal{S}(C)$ minimal. If $H = \{0\}$ then $|A + B| \geq |C| \geq |A \cap B| + |A \cup B| - |\{0\}| = |A| + |B| - 1$ and we are finished. So, we may assume $H \neq \{0\}$ (and proceed toward a contradiction). Since $\mathcal{S}(A + B) = \{0\}$ and $\mathcal{S}(C) = H$, we may choose $a \in A$ and $b \in B$ so that $a+b+H \not\subseteq A+B$. Let $A_1 = A \cap (a+H)$, $A_2 = A \cap (b + H)$, $B_1 = B \cap (b + H)$, and $B_2 = B \cap (a + H)$ and note that $A_1, B_1 \neq \emptyset$. For $i = 1, 2$ let $C_i = C \cup (A_i + B_i)$ and let $H_i = \mathcal{S}(A_i + B_i)$. Observe that if $A_i, B_i \neq \emptyset$, then $H_i = \mathcal{S}(C_i) < H$. The following equation holds for $i = 1$, and it also holds for $i = 2$ if $A_2, B_2 \neq \emptyset$. It follows from the fact that C_i is not a convergent (by the minimality of H), and induction applied to A_i, B_i.

$$\begin{aligned}
|(A\cup B)+H| - |(A\cup B)+H_i| &< (|C|+|H|-|A\cap B|) - (|C_i|+|H_i|-|A\cap B|) \\
&= |H| - |A_i+B_i| - |H_i| \\
&\leq |H| - |A_i+H_i| - |B_i+H_i| \qquad (1)
\end{aligned}$$

If $B_2=\emptyset$, then $|(A\cup B)+H| - |(A\cup B)+H_1| \geq |(a+H)\setminus(A_1+H_1)| = |H|-|A_1+H_1|$ contradicts equation 1 for $i=1$. We get a similar contradiction under the assumption that $A_2=\emptyset$. Thus $A_2, B_2 \neq \emptyset$ and equation 1 holds for $i=1,2$. If $a+H=b+H$, then $A_1=A_2$ and $B_1=B_2$ and we have $|(A\cup B)+H| - |(A\cup B)+H_1| \geq |(a+H)\setminus((A_1\cup B_1)+H_1)| \geq |H|-|A_1+H_1|-|B_1+H_1|$ which contradicts equation 1. Therefore, $a+H\neq b+H$. Our next inequality follows from the observation that the left hand side of equation 1 is nonnegative, and all terms on the right hand side are multiples of $|H_i|$.

$$|H| \geq |A_i| + |B_i| + |H_i| \qquad (2)$$

Let $S=(a+H)\setminus(A_1\cup B_2)$ and $T=(b+H)\setminus(A_2\cup B_1)$, and note that S and T are disjoint. The next equation follows from the fact that $A+B$ is not a convergent (by the minimality of H), and induction applied to A_i, B_i.

$$\begin{aligned}
|H| &\geq |(A\cup B)+H| + |A\cap B| - |C| \\
&\geq |S|+|T|+|A\cup B|+|A\cap B|-|A+B|+|A_i+B_i| \\
&> |S|+|T|+|A_i|+|B_i|-|H_i| \qquad (3)
\end{aligned}$$

Summing the four inequalities obtained by taking equations 2 and 3 for $i=1,2$ and then dividing by two yields $2|H| > |A_1|+|B_2|+|S|+|A_2|+|B_1|+|T|$. However, $a+H=S\cup A_1\cup B_2$ and $b+H=T\cup A_2\cup B_1$. This final contradiction completes the proof. $\square$