# Rado Matroids

Throughout we let $G$ be a bipartite graph with bipartition $(X, Y)$ and we let $M$ denote a matroid on $Y$ with rank function $r$. Our goal is to study a natural matroid which is induced on $X$ by this structure. Since this matroid was first discovered by Rado, we shall call this the *Rado* matroid. We will define this matroid with our first proposition, and then present two theorems about it which generalize classical theorems of Hall and König on bipartite graphs. Indeed, our presentation is done in the setting of bipartite graphs so as to emphasize this connection. As a consequence of our main theorem we also derive the Matroid Union Theorem.

**Proposition 1** *Let $\mathcal{C}$ be the collection of all minimal subsets $S$ of $X$ which satisfy $r(N(S)) < |S|$. Then $\mathcal{C}$ is the set of circuits of a matroid called the* Rado *matroid on $X$.*

*Proof:* If $C_1, C_2 \in \mathcal{C}$ are distinct then we have the following inequality (here the last step follows from the observation that $C_1 \cap C_2$ does not include any member of $\mathcal{C}$)

$$\begin{aligned} |C_1 \cup C_2| + |C_1 \cap C_2| - 2 &= |C_1| + |C_2| - 2 \\ &\geq r(N(C_1)) + r(N(C_2)) \\ &\geq r(N(C_1) \cup N(C_2)) + r(N(C_1) \cap N(C_2)) \\ &\geq r(N(C_1 \cup C_2)) + r(N(C_1 \cap C_2)) \\ &\geq r(N(C_1 \cup C_2)) + |C_1 \cap C_2|. \end{aligned}$$

This implies $r(N(C_1 \cup C_2)) \leq |C_1 \cup C_2| - 2$. If there exists $e \in C_1 \cup C_2$ then setting $C = (C_1 \cup C_2) \setminus \{e\}$ we have

$$r(N(C)) \leq r(N(C_1 \cup C_2)) \leq |C_1 \cup C_2| - 2 = |C| - 1$$

which implies that $C$ must include a member of $\mathcal{C}$. Thus $\mathcal{C}$ obeys the circuit axioms, and this completes the proof. $\square$

Let us remark that in the special case when the original matroid on $Y$ has no dependent sets, the Rado matroid on $X$ is known as a transversal matroid. For transversal matroids, the following proposition is equivalent to Hall's theorem on perfect matchings (since in this case a set $S \subseteq X$ is independent if and only if $|N(R)| \geq |R|$ for all $R \subseteq S$).

**Theorem 2 (Rado)** *A set $S \subseteq X$ is independent if and only if there exists $T \subseteq Y$ so that the subgraph induced by $S \cup T$ has a perfect matching and further $T$ is independent.*

*Proof:* The "if" direction is immediate. For the "only if" direction, we proceed by induction on $|S|$ with the base case $|S| = 0$ holding trivially. First consider the case that $r(N(R)) > |R|$ for every nonempty proper subset $R \subseteq S$. Then, choose $e \in S$ and choose a non-loop $f \in N(e)$ (this is possible since $S$ is independent). Now define a new graph $G' = G - \{e, f\}$ and a new matroid $M' = M/f$ and consider the Rado matroid on $X \setminus \{e\}$ from $G'$ and $M'$. We claim that $S' = S \setminus \{e\}$ is independent in this new matroid. To see this, note that for every $R \subseteq S'$ we have $r_{M'}(N_{G'}(R)) \geq r_M(N_G(R)) - 1 \geq |R|$. Therefore, by induction, we may choose a set $T' \subseteq Y \setminus \{f\}$ so that the graph induced by $S' \cup T'$ has a perfect matching and $T'$ is independent in $M/f$. But then $T = T' \cup \{f\}$ is independent in $M$ and has a perfect matching to $S$.

Next consider the case that $S_0 \subset S$ is nonempty and $r(N(S_0)) = |S_0|$. Now, by induction, we may choose a set $T_0 \subseteq Y$ so that the graph induced by $S_0 \cup T_0$ has a perfect matching and $T_0$ is independent in $M$. Then, we modify $G$ to form $G'$ by deleting the $S_0 \cup T_0$, we define the matroid $M' = M/T_0$ on $Y' = Y \setminus T_0$ and we consider as well the new Rado matroid on $X' = X \setminus S_0$. As before, we claim that $S' = S \setminus S_0$ is independent in this new matroid. To see this, let $R \subseteq S'$ and note that $r_{M'}(N_{G'}(R)) = r_M(N_{G'}(R) \cup T_0) - |T_0| = r_M(N_G(R \cup S_0)) - |S_0| \geq |R|$. So, by induction we may choose a set $T' \subseteq Y'$ so that $G'$ has a perfect matching between $S'$ and $T'$ and so that $T'$ is independent in $M'$. But then $T_0 \cup T'$ has a perfect matching to $S$ and is independent in $M$. $\square$

**Corollary 3 (Matroid Union)** *Let $M_1, M_2, \ldots, M_k$ be matroids on $E$ and let $\mathcal{I}$ be the collection of all $X \subseteq E$ for which there exist disjoint sets $X_1, \ldots, X_k$ so that $X_i$ is independent in $M_i$ and $\cup_{i=1}^k X_i = X$. Then $\mathcal{I}$ is the set of independent sets of a matroid called the* union *of $M_1 \ldots M_k$.*

*Proof:* Let $E_1, E_2, \ldots, E_k$ be disjoint copies of $E$, set $Y = \cup_{i=1}^k E_i$ and call a set $S \subseteq Y$ *independent* if $S \cap E_i$ corresponds to an independent set in $M_i$. It follows immediately from the independent set axioms that these are the independent sets of a matroid on $Y$. Now, construct a graph $G$ with bipartition $(E, Y)$ by connecting every $e \in E$ to each copy of this element. It now follows from the previous theorem that $\mathcal{I}$ is precisely the collection of independent sets of the Rado matroid on $E$. $\square$

König's Theorem gives the following simple characterization for our bipartite graph $G$: The size of a largest matching is equal to the size of a minimum vertex cover. Equivalently, there exists $A \subseteq X$ so that the size of the largest matching is equal to $|X \setminus A| + |N(A)|$. The following theorem gives an analogous result in our more general framework. In this setting, we call a matching in $G$ *independent* if the set of vertices which it covers in $Y$ is independent (from which it follows that the set of vertices it covers in $X$ will also be independent).

**Theorem 4 (Rado)** *There exists $A \subseteq X$ so that the size of the largest independent matching is equal to $|X \setminus A| + r(N(A))$.*

*Proof:* Define the *deficiency* of a set $S \subseteq X$ to be $|S| - r(N(S))$. Let $d$ be the maximum deficiency of a subset of $X$ and note that $d \geq 0$ since $\emptyset$ has deficiency 0. We shall prove that the largest independent matching in $G$ has size $|X| - d$ by proving both inequalities.

To see that there is an independent matching of size at least $|X| - d$ we construct a new graph $G'$ with bipartition $(X, Y')$ from $G$ by adding a set $Z$ of $d$ new vertices to $Y$ (thus forming $Y'$) and then adding an edge between every point in $X$ and every point in $Z$. Then, we modify the matroid $M$ on $Y$ to get a matroid $M'$ on $Y'$ by adding each element in $Z$ as a coloop (so the bases in $M'$ consist of bases in $M$ union with $Z$). Now for every $S \subseteq X$ we have $r_{M'}(N_{G'}(S)) = r_M(N_G(S)) + d$ so $X$ is an independent set in the Rado matroid on $X$ given by $G'$ and $M'$. Thus there exists an independent matching $\mu'$ in $G'$ which covers $X$. Let $\mu$ be obtained from $\mu'$ by removing any edge incident with a point in $Z$. Then $\mu$ is an independent matching in $G$ with size at least $|X| - d$.

To see that every independent matching has size at most $|X| - d$ just consider a particular set $A$ of deficiency $d$. Every independent matching will fail to cover at least $d$ vertices of $A$ so will have size at most $|X| - d$. Now, to complete the proof, observe that the size of the largest independent matching is $|X| - d = |X \setminus A| + |A| - d = |X \setminus A| + r(N(A))$. $\square$

**Corollary 5** *For every $S \subseteq X$ the rank of $S$ is given by*

$$r(S) = \min_{A \subseteq S} |S \setminus A| + r(N(A)).$$

*Proof:* Modify $G$ by deleting all vertices in $X \setminus S$. The rank of $S$ is then equal to the size of the largest independent matching in this new graph. By the previous theorem, this is given by the above formula. $\square$