Restricted Bases

Our goal is to give a proof of the following theorem which implies the existence of a basis of a matroid satisfying certain added restrictions, under the assumption that a suitable fractional basis (a term we define shortly) exists. The key idea in the proof is a pretty recursive process due to Kamal Jain.

Theorem 1 (Király, Lau, Singh) *Let M be a matroid on E, let $x \in \mathbb{R}^E$ be a fractional basis, and let $\mathcal{F}$ be a collection of subsets of E so that every $e \in E$ is contained in at most d members of $\mathcal{F}$. Then there exists a basis B so that every $F \in \mathcal{F}$ satisfies*

$$|B \cap F| \leq \lceil x(F) \rceil + d - 1.$$

This theorem has numerous applications such as the following.

Corollary 2 *Every r-regular r-edge-connected graph has a spanning tree of max degree ≤ 3.*

Sketch of Proof: Let $G = (V, E)$ be such a graph with $|V| = n$ and define the vector $x \in \mathbb{R}^E$ by the rule that $x(e) = \frac{2(n-1)}{nr}$. It follows from the edge-connectivity of G that x is a fractional spanning tree (i.e. a fractional basis for the cycle matroid). Furthermore, for every $v \in V$ the set $\delta(v) = \{e \in E \mid e \sim v\}$ has the property that $x(\delta(v)) = \frac{2(n-1)}{n} \leq 2$. So, setting $\mathcal{F} = \{\delta(v) \mid v \in V\}$ we may apply the theorem to the cycle matroid of G for the vector x, the collection $\mathcal{F}$ and $d = 2$. The result is a spanning tree of G which has maximum degree ≤ 3 as desired. $\square$

The proof of the theorem involves a polyhedral argument which relies on a basic (but important!) fact about systems of linear equations which we give next. For any set of vectors X in Euclidean space we let $rank(X)$ denote the dimension of the subspace spanned by X.

Proposition 3 *Let A be a real matrix indexed by $S \times T$, let $b \in \mathbb{R}^S$, and let P be be the polyhedron defined by the following system of linear equations for $x \in \mathbb{R}^T$*

$$x \geq 0$$
$$Ax \leq b$$

Let y be a vertex of P. For every $s \in S$ let A_s denote the row of A indexed by s, and call s tight *if $A_s y = b_s$. Then we have*

$$|supp(y)| \leq rank(\{A_s \mid s \text{ is tight}\}).$$

Proof: Let A' be the submatrix of A consisting of those rows A_s for which s is tight and suppose (for a contradiction) that $|supp(y)| > rank(A')$. In this case, it follows from dimension considerations that there must exist a nonzero vector $z \in \mathbb{R}^T$ in the nullspace of A' for which $supp(z) \subseteq supp(y)$. Now consider the vector $y' = y + \delta z$ for $\delta \in \mathbb{R}$. For every tight constraint s we have $A_s y' = A_s(y + \delta z) = A_s y = b_s$ so y' will still satisfy all of the tight constraints. It follows from this that there exists $\epsilon > 0$ so that $y' \in P$ whenever $|\delta| < \epsilon$. However, this contradicts the assumption that y is a vertex of P (as we have found a line segment contained in P which has y in its interior). $\square$

Basis Polytope: Let M be a matroid on E with rank function r. We define the *basis polytope of* M, denoted $P(M)$, to be the set of all $x \in \mathbb{R}^E$ which satisfy the following system of constraints.

1. $x \geq 0$.
2. $x(S) \leq r(S)$ for every closed set $S \subseteq E$.
3. $x(E) \geq r(E)$.

Observe that the type 2 constraints imply that every $x \in P(G)$ satisfies $x \leq 1$. Furthermore, the $\{0, 1\}$ valued points in $P(M)$ are precisely the characteristic vectors of bases, so we shall call arbitrary elements of $P(M)$ *fractional bases*. As usual, if $x \in P(M)$ and $x(S) = r(S)$ for some closed set S we say that S is *tight* with respect to x. Our next lemma gives an important dimension restriction on the family of tight sets. Here we let $\mathbb{1}_S$ denote the characteristic vector of a subset $S \subseteq E$.

Lemma 4 *Let M be a loopless nonempty matroid and let $x \in P(M)$, then*

$$rank(\{\mathbb{1}_R \mid R \subseteq E \text{ is tight}\}) \leq r(M).$$

Proof: We proceed by induction on $r(M)$. It is immediate from the definition that $\emptyset$ and E are tight constraints. If there are no other tight constraints, then the equation follows from $r(M) > 0$. Otherwise, let $S \subseteq E$ be a minimal nonempty (closed) set which is tight. If T is any other tight set then we have

$$r(S \cap T) + r(S \cup T) \leq r(S) + r(T) = x(S) + x(T) = x(S \cap T) + x(S \cup T)$$

so it follows that both $S \cap T$ and $S \cup T$ are tight. Since S is minimal, it must be that every tight set T satisfies either $S \subseteq T$ or $S \cap T = \emptyset$. Now, consider the matroid $M' = M/S$ on $E' = E \setminus S$ and the vector $x' = x|_{E'}$. For every $T \subseteq E(M')$ we have

$$r_{M'}(T) = r_M(S \cup T) - r(S) \geq x(S \cup T) - x(S) = x'(T)$$

with equality if and only if $S \cup T$ is tight in M. It follows from this that $x' \in P(M')$. Now, choose a collection of tight sets $T_1, T_2, \ldots, T_k$ in M' relative to x' so that $\{\mathbb{1}_{T_1}, \mathbb{1}_{T_2}, \ldots \mathbb{1}_{T_k}\}$ is a basis of the space $\langle \{\mathbb{1}_T \mid T \subseteq E' \text{ is tight for } x' \text{ (in } M')\}\rangle$. We claim that the set $\{\mathbb{1}_S, \mathbb{1}_{S\cup T_1}, \ldots, \mathbb{1}_{S\cup T_k}\}$ is a basis for the space $\langle \{\mathbb{1}_R \mid R \subseteq E \text{ is tight for } x \text{ (in } M)\}\rangle$. To see this, let $R \subseteq E$ be tight with respect to x. Now $R \cup S$ is tight with respect to x so $R \setminus S$ is tight with respect to x'. Thus we may choose $a_i \in \mathbb{R}$ so that $\mathbb{1}_{R\setminus S} = \sum_{i=1}^k a_i \mathbb{1}_{T_i}$. Since R either contains S or is disjoint from S, it follows that there exists $b \in \mathbb{R}$ so that $\mathbb{1}_R = b\mathbb{1}_S + \sum_{i=1}^k a_i \mathbb{1}_{S\cup T_i}$ thus proving the claim. Now the desired bound follows by induction and the following inequality

$$rank(\{\mathbb{1}_R \mid R \subseteq E \text{ is tight}\}) \leq k + 1 \leq r(M') + 1 \leq r(M). \qquad \Box$$

To help solidify our understanding, let's combine our last two results. First let us note that the following is an equivalent description of the linear system for $P(M)$ (here $x \in \mathbb{R}^E$).

1. $x \geq 0$.
2. $\mathbb{1}_S^\top x \leq r(S)$ for every closed set $S \subseteq E$.
3. $-\mathbb{1}_E^\top x \leq -r(E)$.

We have the constraints $\mathbb{1}_E^\top x \leq r(E)$ and $-\mathbb{1}_E^\top x \leq -r(E)$ which will both be tight. However, in terms of the rank of the set of tight constraints, these two correspond to the same vector $\mathbb{1}_E$. Therefore, by the previous lemma and Proposition 3, every vertex $x \in P(M)$ satisfies $|supp(x)| \leq r(M)$. Since $x \leq 1$ we also have $r(M) = x(E) \leq |supp(x)|$. This yields the following theorem of Edmonds.

Theorem 5 (Edmonds) *Every vertex of $P(M)$ is the characteristic vector of a basis.*

Returning to our central purpose, we shall require the following technical lemma before we can prove the main theorem.

Lemma 6 *Let M be a matroid on E let $x \in \mathbb{R}^E$ satisfy $0 < x(e) < 1$ for every $e \in E$ and $x(E) = r(M)$. Let $\mathcal{F}$ be a collection of subsets of E and assume the following conditions are satisfied.*

- *Every $F \in \mathcal{F}$ satisfies $x(F) \in \mathbb{Z}$.*
- *Every $e \in E$ is contained in at most d members of $\mathcal{F}$.*
- *$|\mathcal{F}| \geq r^*(M)$.*

Then either there exists $F \in \mathcal{F}$ with $x(F) \geq |F| - d + 1$ or the following holds: Every element in E is covered exactly d times by members of $\mathcal{F}$, every $F \in \mathcal{F}$ satisfies $x(F) = |F| - d$, and $|\mathcal{F}| = r^*(M)$.

Proof: For every $e \in E$ let d_e be the number of sets in $\mathcal{F}$ which contain e. We may assume that $x(F) < |F| - d + 1$ for every $F \in \mathcal{F}$ (otherwise we are finished) and then by the integrality of $x(F)$ we have $x(F) \leq |F| - d$ for every $F \in \mathcal{F}$. This yields

$$\sum_{F \in \mathcal{F}} x(F) \leq \sum_{F \in \mathcal{F}} (|F| - d) = \sum_{F \in \mathcal{F}} |F| - d|\mathcal{F}| \leq \sum_{F \in \mathcal{F}} |F| - d \cdot r^*(M)$$

and rearranging gives the first inequality in the following chain.

$$d \cdot r^*(M) \leq \sum_{F \in \mathcal{F}} (|F| - x(F)) = \sum_{e \in E} d_e \, (1 - x(e)) \leq d \sum_{e \in E} (1 - x(e)) = d \cdot r^*(M).$$

Thus all inequalities in the above equations are tight, so $d_e = d$ for every $e \in E$ from the second equation, and $x(F) = |F| - d$ for every $F \in \mathcal{F}$ and $|\mathcal{F}| = r^*(M)$ from the first. $\square$

Proof of Theorem 1: For every $F \in \mathcal{F}$ let $q_F = \lceil x(F) \rceil$. Our proof will proceed by constructing a sequence of nonempty polyhedra $Q_0, Q_1, \ldots$ where Q_i is defined by the following system using sets $E_i^0, E_i^1 \subseteq E$ and $\mathcal{F}_i \subseteq \mathcal{F}$.

1. $y \in P(M)$.
2. $y(F) \leq q_F$ for every $F \in \mathcal{F}_i$.
3. $y(e) = 0$ for every $e \in E_i^0$.
4. $y(e) = 1$ for every $e \in E_i^1$.

Our sequence will start with Q_0 which is defined by setting $E_0^0 = E_0^1 = \emptyset$ and $\mathcal{F}_0 = \mathcal{F}$. Note that $x \in Q_0$, so in particular, $Q_0 \neq \emptyset$. We shall continue the process until we obtain a Q_i which contains an integral point. At each step Q_{i+1} will be obtained from Q_i by either adding a single new constraint of type 3 or 4 or by removing a single constraint of type 2. To control this process, we shall only permit the removal of the constraint F from $\mathcal{F}_i$ when the following condition is satisfied:

$$|F \setminus E_i^0| \leq q_F + d - 1 \tag{1}$$

Since the sets $E_0^0, E_1^0, \ldots$ form an increasing chain, and every point in $P(M)$ is ≤ 1, every point $z \in Q_j$ for $j > i$ will satisfy $z(F) \leq |F \setminus E_i^0| \leq q_F + d - 1$ so no such point can violate the constraint F by more than $d - 1$. Therefore, if we can continue this process until some Q_i contains an integral point, say $z \in \mathbb{Z}^E$, this vector will be the characteristic vector of a base B of M which satisfies $|B \cap F| = z(F) \leq q_F + d - 1$ which is all that is required. Thus, to complete the proof, it suffices to prove that we can construct this sequence of polyhedra. So, we shall assume that Q_i is nonempty and does not contain an integral point, and show how to construct Q_{i+1}.

Let z be a vertex of the polyhedron Q_i. If $z(e) = 0$ for some $e \notin E_i^0$ or $z(e) = 1$ for some $e \notin E_i^1$ then we may obtain Q_{i+1} by adding this equation as a new constraint (this new polyhedron will still contain z so will be nonempty). Thus, we may assume that $E' = E \setminus (E_i^0 \cup E_i^1)$ satisfies $0 < z(e) < 1$ for every $e \in E'$. Define the collection

$$\mathcal{H} = \{H \in \mathcal{F}_i \mid z(H) = q_H\}.$$

Now, roughly speaking, our plan will be to use Proposition 3 to show that there are many tight constraints in $\mathcal{H}$ and then to use the previous lemma to show that one of these constraints satisfies equation 1. To do this, it will be helpful to consider the matroid $M' = M/E_i^1 \setminus E_i^0$, together with the polyhedron Q_i' given by the following system for $y' \in \mathbb{R}^{E'}$

1. $y' \in P(M')$

2. $y'(F \cap E') \leq q_F - |F \cap E_i^1|$ for every $F \in \mathcal{F}$

Now, $y' \in P(M')$ if and only if the vector $y \in \mathbb{R}^E$ given by

$$y(e) = \begin{cases} y'(e) & e \in E' \\ 0 & e \in E_i^0 \\ 1 & e \in E_i^1 \end{cases}$$

lies in $P(M)$. Geometrically, $P(M')$ is isomorphic to the face of $P(M)$ obtained by restricting to the affine space with value 0 on E_i^0 and 1 on E_i^1. It follows from this that Q_i' is isomorphic to a face of Q_i. Furthermore, setting $z' = z|_{E'}$ we have that z' is a vertex of Q_i' and $\mathcal{H}$ is precisely the set of constraints of type 2 which are tight for z'. Therefore, by Proposition 3 we have:

$$|E'| \leq rank\left(\{\mathbb{1}_S \mid S \subseteq E' \text{ is tight for } z'\} \cup \{\mathbb{1}_{H \cap E'} \mid H \in \mathcal{H}\}\right) \tag{2}$$

It follows from Lemma 4 that $rank\left(\{\mathbb{1}_S \mid S \subseteq E' \text{ is tight for } z'\}\right) \leq r(M')$. Thus, setting $\mathcal{H}' = \{H \cap E' \mid H \in \mathcal{H}\}$, we have $|\mathcal{H}'| \geq rank(\{\mathbb{1}_{H \cap E'} \mid H \in \mathcal{H}\}) \geq |E'| - r(M') = r^*(M')$. Thus, the previous lemma applies (nontrivially) to the matroid M', the vector z' and the collection $\mathcal{H}'$. If there exists $H' \in \mathcal{H}'$ with $z'(H') \geq |H'| - d + 1$ then choose $H \in \mathcal{H}$ with $H \cap E' = H'$ and observe that

$$q_H = z(H) = |H \cap E_i^1| + z'(H') \geq |H \cap E_i^1| + |H'| - d + 1 = |H \setminus E_i^0| - d + 1$$

so we may safely obtain $\mathcal{Q}_{i+1}$ by removing the constraint corresponding to H. Otherwise, it follows from the previous lemma that $|\mathcal{H}'| = r^*(M')$ and every element in M' is covered by exactly d members of $\mathcal{H}'$. But then, summing all of the vectors $\mathbb{1}_{H \cap E'}$ over all $H \in \mathcal{H}$ yields the vector $d\mathbb{1}_{E'}$. Since E' is tight relative to z' this gives us

$$rank\left(\{\mathbb{1}_S \mid S \subseteq E' \text{ is tight for } z'\} \cup \{\mathbb{1}_{H \cap E'} \mid H \in \mathcal{H}\}\right) \leq r(M') + r^*(M') - 1 = |E'| - 1$$

which contradicts Equation 2. This completes the proof. $\square$