Hoffman's Bound for Vector Colouring

The goal of this note is to prove the following theorem of Bilu which gives a spectral lower bound on the vector chromatic number (which we denote by χ_v). This improves upon a classic bound of Hoffman which has the same statement except with χ in place of χ_v.

Theorem 1 (Bilu) *Let $G = (V, E)$ be a graph with $V = \{1, 2, \ldots, n\}$, let $W \neq 0$ be a nonnegative symmetric $n \times n$ matrix with $W_{i,j} = 0$ whenever $ij \notin E$, and let $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$ be the eigenvalues of W. Then*

$$\chi_v(G) \geq 1 - \frac{\lambda_n}{\lambda_1}.$$

We begin with a little notation followed by a lemma. Throughout we treat elements of $\mathbb{R}^n$ as column vectors. We write $B \succeq 0$ if B is a positive semidefinite matrix, and if A, B are $n \times n$ matrices, their dot product is $A \cdot B = tr(AB^\top)$.

Lemma 2 *Let A be a real symmetric matrix and let λ_n be its smallest eigenvalue. Then*

$$\lambda_n = \min_{v_1,\ldots,v_n \in \mathbb{R}^n} \frac{\sum_{i=1}^n \sum_{j=1}^n A_{i,j}(v_i^\top v_j)}{\sum_{i=1}^n ||v_i||_2^2}.$$

Proof: It follows from the Rayleigh-Ritz characterization that

$$\lambda_n = \min_{x \in \mathbb{R}^n : ||x||^2 = 1} x^\top A x = \min_{x \in \mathbb{R}^n : tr(xx^\top) = 1} A \cdot xx^\top.$$

Now, matrices of the form $xx^\top$ are precisely the positive semidefinite matrices of rank 1. Thus, λ_n is also equal to the minimum of $A \cdot B$ where B ranges over all positive semidefinite matrices of rank 1 and trace 1. However, every positive semidefinite matrix is a nonnegative combination of rank 1 positive semidefinite matrices, so this rank restriction is superfluous. Thus,

$$\lambda_n = \min_{B \succeq 0 : tr(B) = 1} A \cdot B. \tag{1}$$

Every positive semidefinite matrix B is the Gram matrix of a collection of vectors, so we may choose $v_1, \ldots, v_n \in \mathbb{R}^n$ so that $B_{i,j} = v_i^\top v_j$. Now $tr(B) = 1$ is equivalent to $\sum_{i=1}^n ||v_i||_2^2 = 1$ and the result follows immediately from this observation and (1). □

Proof of Theorem 1: Let $t = \chi_v(G)$ and choose unit vectors $u_1, \ldots, u_n \in \mathbb{R}^n$ so that $u_i^\top u_j \leq -\frac{1}{t-1}$ whenever $ij \in E$. Next, choose an eigenvector $\alpha \in \mathbb{R}^n$ of W with eigenvalue λ_1 and note that $\alpha \geq 0$ (by Perron-Frobenius). Finally, for $1 \leq i \leq n$ set $v_i = \alpha_i u_i$. Since $W_{i,j} = 0$ whenever $u_i^\top u_j > -\frac{1}{t-1}$, the lemma gives us

$$\begin{aligned}
\lambda_n &\leq \frac{\sum_{i=1}^n \sum_{j=1}^n W_{i,j}(v_i^\top v_j)}{\sum_{i=1}^n ||v_i||_2^2} \\
&\leq -\frac{1}{t-1} \cdot \frac{\sum_{i=1}^n \sum_{j=1}^n W_{i,j}\alpha_i\alpha_j}{\sum_{i=1}^n \alpha_i^2} \\
&= -\frac{1}{t-1} \frac{\alpha^\top W \alpha}{||\alpha||^2} \\
&= -\frac{\lambda_1}{t-1}.
\end{aligned}$$

Thus $\chi_v(G) = t \geq 1 - \frac{\lambda_1}{\lambda_n}$ as claimed. $\square$