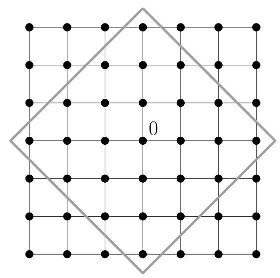
Pretty Theorems on Vertex Transitive Graphs

Growth

For a graph G a vertex x and a nonnegative integer n we let $B(x, n)$ denote the ball of radius n around x (i.e. the set $\{u \in V(G) : dist(u, v) \leq n\}$. If G is a vertex transitive graph then $|B(x, n)| = |B(y, n)|$ for any two vertices x, y and we denote this number by $f(n)$.

Example: If $G = Cayley(\mathbb{Z}^2, \{(0, \pm 1), (\pm 1, 0)\})$ then $f(n) = (n + 1)^2 + n^2$.



$f(3) = |B(0, 3)| = (1 + 3 + 5 + 7) + (5 + 3 + 1) = 4^2 + 3^2$

Our first result shows a property of the function f which is a relative of log concavity.

Theorem 1 (Gromov) *If G is vertex transitive then $f(n)f(5n) \leq f(4n)^2$*

Proof: Choose a maximal set Y of vertices in $B(u, 3n)$ which are pairwise distance $\geq 2n + 1$ and set $y = |Y|$. The balls of radius n around these points are disjoint and are contained in $B(u, 4n)$ which gives us $yf(n) \leq f(4n)$. On the other hand, the balls of radius $2n$ around the points in Y cover $B(u, 3n)$, so the balls of radius $4n$ around these points cover $B(u, 5n)$, giving us $yf(4n) \geq f(5n)$. Combining our two inequalities yields the desired bound. □

Isoperimetric Properties

Here is a classical problem: Given a small loop of string in the plane, arrange it to maximize the enclosed area. Perhaps not surprisingly, the best you can do is to arrange your string as a circle. So, the real problem amounts to proving that any region in the plane with area greater than that of a circle also has a larger boundary. Such inequalities (relating area to boundary) are called isoperimetric inequalities, and are of great interest in a variety of contexts, in particular for vertex transitive graphs.

The *diameter* of a subset of vertices X, denoted $diam(X)$ is the maximum distance between a pair of points in X. For a set X of vertices we let $\partial(X) = \{v \in V(G) \setminus X : v \text{ is adjacent to a point in } X\}$.

Example: We again consider the Cayley graph on $\mathbb{Z}^2$ from the previous subsection. If we take $X = B(0,n)$ then $|X| = (n+1)^2 + n^2$ and $|\partial(X)| = |B(0,n+1) \setminus B(0,n)| = (n+2)^2 - n^2 = 2n+4$, so just as in the plane, the size of the boundary is proportional to n while the size of the set (i.e. area) is proportional to n^2.

Theorem 2 (Babai Szegedy) *If G is vertex transitive and $X \subseteq V(G)$ with $diam(X) < diam(G)$ then*

$$\frac{|\partial(X)|}{|X|} \geq \frac{1}{diam(X)+1}$$

Proof: Let N denote the number of geodesic paths of length $d = diam(X) + 1$ which pass through an arbitrary vertex of G (since G is vertex transitive this number is the same for any two vertices). Let $\mathcal{P}$ denote the set of geodesic paths of length d which intersect the set X. The key observation for this proof is just that every $P \in \mathcal{P}$ meets the set $\partial(X)$. It follows instantly from this that

$$|\mathcal{P}| \leq N|\partial(X)|.$$

On the other hand, we have

$$N|X| = \sum_{P \in \mathcal{P}} |V(P) \cap X| \leq d|\mathcal{P}|$$

(here there is a small subtlety that every path in $\mathcal{P}$ has $d+1$ points, at most d of which are in X). Combining our two inequalities yields the result. $\square$

Long Cycles

There are only 4 connected vertex transitive graphs with ≥ 3 vertices known not to have a Hamiltonian cycle: *Petersen's Graph*, *Coxeter's Graph*, and the graphs obtained form these by truncation (blowing up each vertex to a triangle). In particular, this leaves open the following famous question.

Problem 3 *Does every connected vertex transitive graph have a Hamiltonian path?*

Our next theorem shows that vertex transitive graphs must have long cycles (although what we can show is still well short of Hamiltonicity). But first we need a little notation and a couple simple facts. If the group Γ acts transitively on the set X and $x \in X$ we let $\Gamma_x = \{g \in \Gamma : g(x) = x\}$ and we call this the *stabilizer* of x. The orbit stabilizer theorem tells us that $|\Gamma| = |X| \cdot |\Gamma_x|$.

Lemma 4 *Let Γ act transitively on the set X and let $S \subseteq X$. If $|S \cap g(S)| \geq c$ for every $g \in \Gamma$ then $|S| \geq \sqrt{c|X|}$.*

Proof: Let N be the number of pairs (g, x) so that $g \in \Gamma$, $x \in X$ and $x \in S \cap g(S)$. For every $x \in S$ we have that (x, g) will contribute to N if and only if $g^{-1}(x) \in S$. Since there are exactly $|\Gamma_x|$ ways to map g to a given point, it follows that there are exactly $|S| \cdot |\Gamma_x|$ terms containing x which contribute to N. This gives us

$$|S|^2 \cdot |\Gamma_x| = N$$

On the other hand, by assumption, for every group element g we have that there exist at least c points x so that (x, g) contributes to N. This gives us

$$N \geq c|\Gamma|$$

Combining our inequalities gives us $|S|^2 \cdot |\Gamma_x| \leq c|\Gamma|$ and then dividing through by $|\Gamma_x|$ (and applying the Orbit Stabilizer Theorem yields $|S|^2 \leq c|X|$ as desired. $\square$

Theorem 5 (Babai) *If $G = (V, E)$ is a connected vertex transitive graph with $n = |V|$ then G has a cycle of length $\geq \sqrt{3n}$*

Proof: Every finite connected vertex transitive graph is 2-connected, and must be 3-connected if it is not a cycle (a fact we leave without proof here). Thus, we may assume that G is 3-connected. Now, choose a longest cycle C of G and consider the set $S = V(C) \subseteq V$. It follows from the 3-connectivity of G that $|S \cap g(S)| \geq 3$ for every automorphism g. So, the previous lemma gives us $|S| \geq \sqrt{3n}$ as desired. $\square$