Combinatorial Optimization: Packing and Covering

Gérard Cornuéjols
Carnegie Mellon University

July 2000

Preface

The integer programming models known as set packing and set covering have a wide range of applications, such as pattern recognition, plant location and airline crew scheduling. Sometimes, due to the special structure of the constraint matrix, the natural linear programming relaxation yields an optimal solution that is integer, thus solving the problem. Sometimes, both the linear programming relaxation and its dual have integer optimal solutions. Under which conditions do such integrality properties hold? This question is of both theoretical and practical interest. Min-max theorems, polyhedral combinatorics and graph theory all come together in this rich area of discrete mathematics. In addition to min-max and polyhedral results, some of the deepest results in this area come in two flavors: "excluded minor" results and "decomposition" results. In these notes, we present several of these beautiful results. Three chapters cover min-max and polyhedral results. The next four cover excluded minor results. In the last three, we present decomposition results. We hope that these notes will encourage research on the many intriguing open questions that still remain. In particular, we state 18 conjectures. For each of these conjectures, we offer \$5000 as an incentive for the first correct solution or refutation before December 2020.

Contents

Acknowledgements: Many thanks to Jon Lee, Kristina Vušković and Carl Lee for encouraging me to write this monograph and to François Margot, Beth Novick, Doug West, Michele Conforti, Thomas Zaslavsky and Andre Kundgen for their comments on the first draft. These notes are based on a series of ten lectures delivered at a regional

CBMS-NSF Conference at the University of Kentucky in May 1999. The work was supported in part by NSF grant DMI-9802773 and ONR grant N00014-97-1-0196.

Chapter 1

Clutters

A *clutter* $\mathcal{C}$ is a pair (V, E), where V is a finite set and E is a family of subsets of V none of which is included in another. The elements of V are the *vertices* of $\mathcal{C}$ and those of E are the *edges*. For example, a simple graph (V, E) (no multiple edges or loops) is a clutter. We refer to West [208] for definitions in graph theory. In a clutter, a *matching* is a set of pairwise disjoint edges. A *transversal* is a set of vertices that intersects all the edges. A clutter is said to *pack* if the maximum cardinality of a matching equals the minimum cardinality of a transversal. This terminology is due to Seymour 1977. Many min-max theorems in graph theory can be rephrased by saying that a clutter packs. We give three examples. The first is König's theorem.

Theorem 1.1 (König [130]) *In a bipartite graph, the maximum cardinality of a matching equals the minimum cardinality of a transversal.*

As a second example, let s and t be distinct nodes of a graph G. Menger's theorem states that the maximum number of pairwise edge-disjoint st-paths in G equals the minimum number of edges in an st-cut (see West [208] Theorem 4.2.18). Let $\mathcal{C}_1$ be the clutter whose vertices are the edges of G and whose edges are the st-paths of G (Following West's terminology [208], paths and cycles have no repeated nodes). We call $\mathcal{C}_1$ the *clutter of st-paths.* Its transversals are the st-cuts. Thus Menger's theorem states that the clutter of st-paths packs.

Interestingly, some difficult results and famous conjectures can be rephrased by saying that certain clutters pack. As a third example, consider the four color theorem [2] stating that every planar graph is 4-colorable. Tait [190] showed that this theorem is equivalent to the following statement: Every simple 2-connected cubic planar graph G is 3-edge-colorable (see West [208] Theorem 7.3.3). Let $\mathcal{C}_2 = (V(\mathcal{C}_2), E(\mathcal{C}_2))$ be the clutter whose vertices are the maximal matchings of G and whose edges are indexed by the edges of G, with $S_e \in E(\mathcal{C}_2)$ if and only if $S_e = \{M \in V(\mathcal{C}_2) : \; e \in M\}$. In these notes, a *maximal* set in a given family refers to an inclusion-maximal set, whereas a *maximum* set refers to a set of maximum cardinality. We make the same distinction between *minimal* and *minimum*. We leave it as an exercise to check that, in a cubic graph, $\mathcal{C}_2$ packs if and only if G is 3-edge-colorable. Therefore, the four color theorem is equivalent to stating that $\mathcal{C}_2$ packs for simple 2-connected cubic planar graphs. The smallest simple 2-connected cubic graph that is not 3-edge-colorable is the Petersen graph (see Figure 1.1). Tutte [203] conjectured that every simple 2-connected cubic graph that is not 3-edge-colorable (i.e. $\mathcal{C}_2$ does not pack) is contractible to the Petersen graph. (Graph G is *contractible* to graph H if H can be obtained from G by a sequence of edge contractions and edge deletions. *Contracting* edge $e = uv$ is the operation of replacing u and v by a single node whose incident edges are the edges other than e that were incident to u or v. *Deleting* e is the operation of removing e from the graph.) Since the Petersen graph is not planar, the four color theorem is a special case of Tutte's conjecture. Tutte's conjecture was proved recently by Robertson, Sanders, Seymour and Thomas ([165], [169], [170]). A more general conjecture of Conforti and Johnson [55] is still open (see Section 1.3.5). This indicates that a full understanding of the clutters that pack must be extremely difficult. More restricted notions are amenable to beautiful theories, while still containing rich classes of examples. In this chapter, we introduce several such concepts and examples.

Exercise 1.2 *In a cubic graph G, show that $\mathcal{C}_2$ packs if and only if G is 3-edge-colorable.*

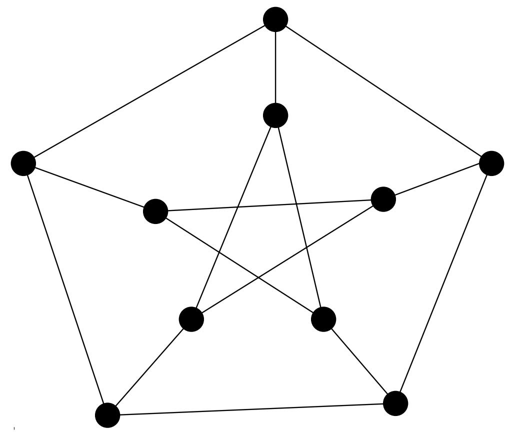

Figure 1.1: The Petersen graph.

1.1 MFMC Property and Idealness

We define a *clutter* $\mathcal{C}$ to be a family $E(\mathcal{C})$ of subsets of a finite ground set $V(\mathcal{C})$ with the property that $S_1 \not\subseteq S_2$ for all distinct $S_1, S_2 \in E(\mathcal{C})$. $V(\mathcal{C})$ is called the set of *vertices* and $E(\mathcal{C})$ the set of *edges* of $\mathcal{C}$. A clutter is *trivial* if it has no edge or if it has the empty set as unique edge. Clutters are also called *Sperner families* in the literature.

Given a nontrivial clutter $\mathcal{C}$, we define $M(\mathcal{C})$ to be a 0,1 matrix whose columns are indexed by $V(\mathcal{C})$, whose rows are indexed by $E(\mathcal{C})$ and where $m_{ij} = 1$ if and only if the vertex corresponding to column j belongs to the edge corresponding to row i. In other words, the rows of $M(\mathcal{C})$ are the characteristic vectors of the sets in $E(\mathcal{C})$. Note that the definition of $M(\mathcal{C})$ is unique up to permutation of rows and permutation of columns. Furthermore, $M(\mathcal{C})$ contains no dominating row, since $\mathcal{C}$ is a clutter (A vector $r \in F$ is said to be *dominating* if there exists $v \in F$ distinct from r such that $r \geq v$). A 0,1 matrix containing no dominating rows is called a *clutter matrix*. Given any 0,1 clutter matrix M, let $\mathcal{C}(M)$ denote the clutter such that $M(\mathcal{C}(M)) = M$.

Let $M \neq 0$ be a 0,1 clutter matrix and consider the following pair of dual linear programs.

$$\min\{wx : x \geq 0,\ Mx \geq \mathbf{1}\} \tag{1.1}$$

$$= \max\{y\mathbf{1} : y \geq 0,\ yM \leq w\} \tag{1.2}$$

Here x and $\mathbf{1}$ are column vectors while w and y are row vectors. $\mathbf{1}$ denotes a vector all of whose components are equal to 1.

Definition 1.3 *Clutter $\mathcal{C}(M)$* packs *if both (1.1) and (1.2) have optimal solution vectors x and y that are integral when $w = \mathbf{1}$.*

Definition 1.4 *Clutter $\mathcal{C}(M)$ has the* packing property *if both (1.1) and (1.2) have optimal solution vectors x and y that are integral for all vectors w with components equal to $0, 1$ or $+\infty$.*

Definition 1.5 *Clutter $\mathcal{C}(M)$ has the* Max Flow Min Cut *property (or MFMC property) if both (1.1) and (1.2) have optimal solution vectors x and y that are integral for all nonnegative integral vectors w.*

Clearly, the MFMC property for a clutter implies the packing property which itself implies that the clutter packs. Conforti and Cornuéjols [41] conjectured that, in fact, the MFMC property and the packing property are identical. This conjecture is still open.

Conjecture 1.6 *A clutter has the MFMC property if and only if it has the packing property.*

Definition 1.7 *Clutter $\mathcal{C}(M)$ is* ideal *if (1.1) has an optimal solution vector x that is integral for all $w \geq 0$.*

The notion of idealness is also known as the *width-length property* (Lehman [133]), the *weak Max Flow Min Cut property* (Seymour [183]) or the *$\mathcal{Q}_+$-MFMC property* (Schrijver [172]). It is easy to show that the MFMC property implies idealness. Indeed, if (1.1) has an optimal solution vector x for all nonnegative integral vectors w, then (1.1) has an optimal solution x for all nonnegative rational vectors w and, since the rationals are dense in the reals, for all $w \geq 0$. In fact, the packing property implies idealness.

Theorem 1.8 *If a clutter has the packing property, then it is ideal.*

This follows from a result of Lehman [133] that we will prove in Chapter 4 (see Theorem 4.1 and Exercise 4.8).

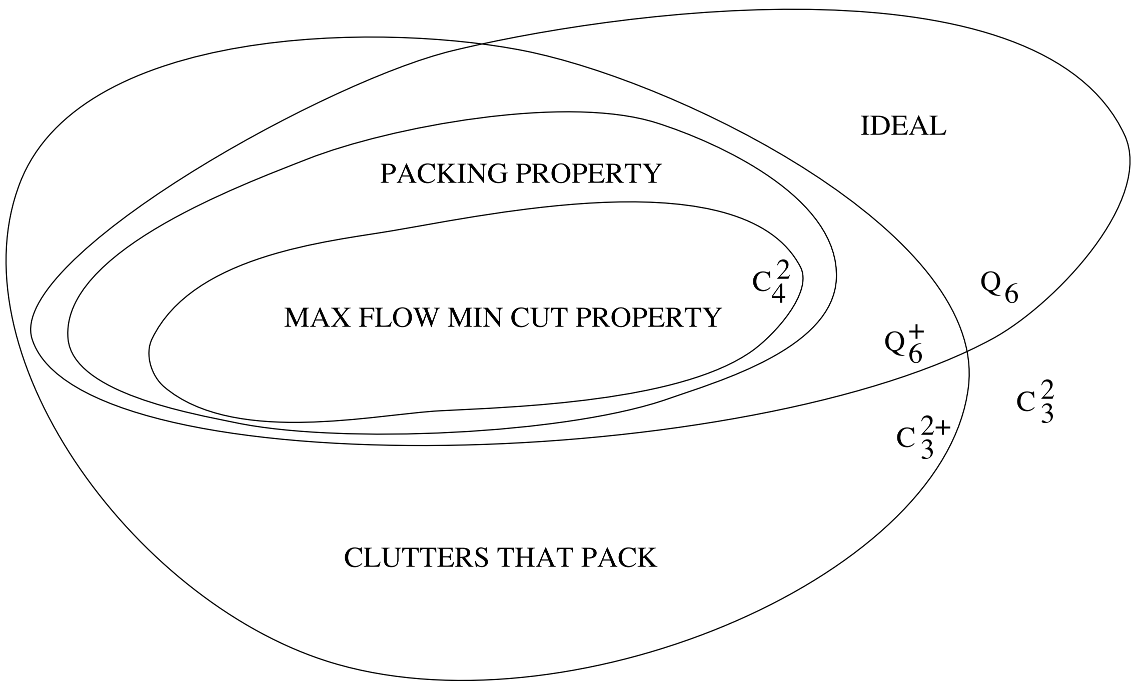

Figure 1.2: Classes of clutters.

Exercise 1.9 *Let* $Q_6 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$, $C_3^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ *and* $C_4^2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. *For an* $m \times n$ *0,1 matrix* M, *let* M^+ *denote the* $m \times (n+1)$ *matrix obtained from* M *by adding the column vector* $\mathbf{1}$. *Find which of the clutters* $\mathcal{C}(Q_6)$, $\mathcal{C}(Q_6^+)$, $\mathcal{C}(C_3^2)$, $\mathcal{C}(C_3^{2+})$, $\mathcal{C}(C_4^2)$ *pack, which have the packing property, which have the MFMC property and which are ideal. See Figure 1.2 for a hint.*

Clearly, $\mathcal{C}(M)$ is ideal if and only if $P = \{x \geq 0 : Mx \geq \mathbf{1}\}$ is an *integral polyhedron*, that is, P has only integral extreme points. Equivalently, $\mathcal{C}$ is ideal if and only if

$$\begin{aligned} x(S) &\geq 1 \quad \text{for all} \;\; S \in E(\mathcal{C}) \\ x &\geq 0 \end{aligned}$$

is an integral polyhedron, where $x(S)$ denotes $\sum_{i \in S} x_i$.

A linear system $Ax \geq b$ is *Totally Dual Integral* (TDI) if the linear program $\min\{wx : \; Ax \geq b\}$ has an integral optimal dual solution y for every integral w for which the linear program has a finite optimum. Edmonds and Giles [81] showed that, if $Ax \geq b$ is TDI and b is integral, then $P = \{x : Ax \geq b\}$ is an integral polyhedron. The interested reader can find the proof of the Edmonds-Giles theorem in Schrijver [173] pages 310–311, or Nemhauser and Wolsey [146] pages 536–537. It follows that $\mathcal{C}(M)$ has the MFMC property if and only if (1.2) has an optimal integral solution y for all nonnegative integral vectors w.

Definition 1.10 *Let* k *be a positive integer. Clutter* $\mathcal{C}(M)$ *has the* $1/k$ -MFMC property *if it is ideal and, for all nonnegative integral vectors* w, *the linear program (1.2) has an optimal solution vector* y *such that* ky *is integral.*

When $k = 1$, this definition reduces to the MFMC property. If $\mathcal{C}(M)$ has the $1/k$-MFMC property, then it also has the $1/q$-MFMC property for every integer q that is a multiple of k.

For convenience, we say that trivial clutters have all the above properties: MFMC, ideal, etc.

The min-max equation (1.1)=(1.2) has a close max-min relative

$$\begin{aligned} &\max\{wx : x \geq 0,\ Mx \leq \mathbf{1}\} \\ =&\min\{y\mathbf{1} : y \geq 0,\ yM \geq w\} \end{aligned}$$

discussed in Chapter 3.

1.2 Blocker

The *blocker* $b(\mathcal{C})$ of a clutter $\mathcal{C}$ is the clutter with $V(\mathcal{C})$ as vertex set and the minimal transversals of $\mathcal{C}$ as edge set. That is, $E(b(\mathcal{C}))$ consists of the minimal members of $\{B \subseteq V(\mathcal{C}) : |B \cap A| \geq 1 \text{ for all } A \in E(\mathcal{C})\}$. In other words, the rows of $M(b(\mathcal{C}))$ are the minimal 0,1 vectors x^T such that x belongs to the polyhedron $\{x \geq 0 :\ M(\mathcal{C})x \geq \mathbf{1}\}$.

Example 1.11 *Let G be a graph and s, t be distinct nodes of G. If $\mathcal{C}$ is the clutter of st-paths, then $b(\mathcal{C})$ is the clutter of minimal st-cuts.*

Exercise 1.12 *Show that the blocker of a trivial clutter is a trivial clutter.*

Edmonds and Fulkerson [80] observed that $b(b(\mathcal{C})) = \mathcal{C}$. Before proving this property, we make the following remark.

Remark 1.13 *Let $\mathcal{H}$ and $\mathcal{K}$ be two clutters defined on the same vertex set. If*

(i) every edge of $\mathcal{H}$ contains an edge of $\mathcal{K}$ and

(ii) every edge of $\mathcal{K}$ contains an edge of $\mathcal{H}$,

then $\mathcal{H} = \mathcal{K}$.

Exercise 1.14 *Prove Remark 1.13.*

Theorem 1.15 *If $\mathcal{C}$ is a clutter, then $b(b(\mathcal{C})) = \mathcal{C}$.*

Proof: Let A be an edge of $\mathcal{C}$. The definition of $b(\mathcal{C})$ implies that $|A \cap B| \geq 1$, for every edge B of $b(\mathcal{C})$. So A is a transversal of $b(\mathcal{C})$, i.e. A contains an edge of $b(b(\mathcal{C}))$.

Now let A be an edge of $b(b(\mathcal{C}))$. We claim that A contains an edge of $\mathcal{C}$. Suppose otherwise. Then $V(\mathcal{C}) - A$ is a transversal of $\mathcal{C}$ and therefore it contains an edge B of $b(\mathcal{C})$. But then $A \cap B = \emptyset$ contradicts the fact that A is an edge of $b(b(\mathcal{C}))$. So the claim holds.

Now the theorem follows from Remark 1.13. $\square$

Two 0,1 matrices of the form $M(\mathcal{C})$ and $M(b(\mathcal{C}))$ are said to form a *blocking pair*. The next theorem is an important result due to Lehman [132]. It states that, for a blocking pair A, B of 0,1 matrices, the polyhedron P defined by

$$Ax \geq \mathbf{1} \tag{1.3}$$
$$x \geq 0 \tag{1.4}$$

is integral if and only if the polyhedron Q defined by

$$Bx \geq \mathbf{1} \tag{1.5}$$
$$x \geq 0 \tag{1.6}$$

is integral. The proof of this result uses the following remark.

Remark 1.16

(i) The rows of B are exactly the 0,1 extreme points of P.

(ii) If an extreme point x of P satisfies $x^T \geq \lambda^T B$ where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then x is a 0,1 extreme point of P.

Proof: (i) follows from the fact that the rows of B are the minimal 0,1 vectors in P.

To prove (ii), note that x is an extreme point of $P_I = \{\chi : \ \chi^T \geq \lambda^T B$ where $\lambda_i \geq 0$ and $\sum \lambda_i = 1\}$ for otherwise x would be a convex combination of distinct $x^1, x^2 \in P_I$ and, since $P_I \subseteq P$, this would contradict the assumption that x is an extreme point of P. Now (ii) follows by observing that the extreme points of P_I are exactly the rows of B. $\square$

Theorem 1.17 (Lehman [132]) *A clutter is ideal if and only if its blocker is.*

Proof: By Theorem 1.15, it suffices to show that if P defined by (1.3)-(1.4) is integral, then Q defined by (1.5)-(1.6) is also integral.

Let a be an arbitrary extreme point of Q. By (1.5), $Ba \geq \mathbf{1}$, i.e. $a^T x \geq 1$ is satisfied by every x such that x^T is a row of B. Since P is an integral polyhedron, it follows from Remark 1.16(i) that $a^T x \geq 1$ is satisfied by all the extreme points of P. By (1.6), $a \geq 0$. Therefore $a^T x \geq 1$ is satisfied by all points in P. Furthermore, $a^T x = 1$ for some $x \in P$. Now, by linear programming duality, we have

$$1 \quad = \quad \min\{a^T x : \; x \in P\} \quad = \quad \max\{\lambda^T \mathbf{1} : \; \lambda^T A \leq a^T, \; \lambda \geq 0\}.$$

Therefore, by Remark 1.16(ii) applied to Q, a is a 0,1 extreme point of Q. $\square$

Exercise 1.18 *Let Q_6 denote the 4×6 incidence matrix of triangles versus edges of K_4. Describe the blocker of $\mathcal{C}(Q_6)$. Is it ideal? Does it pack? Does it have the MFMC property? Compare with the properties of Q_6 found in Exercise 1.9.*

1.3 Examples

1.3.1 st-Cuts and st-Paths

Consider a digraph (N, A) with $s, t \in N$. Let $\mathcal{C}$ be the clutter where $V(\mathcal{C}) = A$ and $E(\mathcal{C})$ is the family of st-paths.

For any arc capacities $w \in Z_+^A$, the Ford-Fulkerson theorem [84] states that (1.1) and (1.2) both have optimal solutions that are integral: (1.1) is the min cut problem and (1.2) is the max flow problem (a *flow* y is a multiset of st-paths such that each arc $a \in A$ belongs to at most w_a st-paths of y. A *max flow* is a flow containing the maximum number of st-paths). Using the terminology introduced in Definition 1.5, the Ford-Fulkerson theorem states that the clutter $\mathcal{C}$ of st-paths has the MFMC property.

Theorem 1.19 (Ford-Fulkerson [84]) *The clutter $\mathcal{C}$ of st-paths has the MFMC property.*

This result implies that $\mathcal{C}$ is ideal and therefore the polyhedron

$$\{x \in R_+^A : x(P) \geq 1 \text{ for all } st\text{-paths } P\}$$

is integral. Its extreme points are the minimal st-cuts. In the remainder, it will be convenient to refer to minimal st-cuts simply as st-cuts.

As a consequence of Lehman's theorem (Theorem 1.17), the clutter of st-cuts is also ideal, i.e. the polyhedron

$$\{x \in R_+^A : x(C) \geq 1 \text{ for all } st\text{-cuts } C\}$$

is integral. So, minimizing a nonnegative linear function over this polyhedron solves the shortest st-path problem. We leave it as an exercise to show that the clutter of st-cuts has the MFMC property.

Exercise 1.20 *Show that the clutter of st-cuts packs by using graph theoretic arguments. Then show that the clutter of st-cuts has the MFMC property.*

In a network, the product of the minimum number of edges in an st-path by the minimum number of edges in an st-cut is at most equal to the total number of edges in the network. This width-length inequality can be generalized to any nonnegative edge lengths ℓ_e and widths w_e: the minimum length of an st-path times the minimum width of an st-cut is at most equal to the scalar product $\ell^T w$. This width-length inequality was observed by Moore and Shannon [145] and Duffin [78]. A length and a width can be defined for any clutter and its blocker. Interestingly, Lehman [132] showed that the width-length inequality can be used as a characterization of idealness.

Theorem 1.21 (Width-length inequality, Lehman [132]) *For a clutter $\mathcal{C}$ and its blocker $b(\mathcal{C})$, the following statements are equivalent.*

- *$\mathcal{C}$ and $b(\mathcal{C})$ are ideal;*
- $\min\{w(C) : C \in E(\mathcal{C})\} \times \min\{\ell(D) : D \in E(b(\mathcal{C}))\} \leq w^T\ell$ *for all* $\ell, w \in R_+^n$.

Proof: Let $A = M(\mathcal{C})$ and $B = M(b(\mathcal{C}))$ be the blocking pair of 0,1 matrices associated with $\mathcal{C}$ and $b(\mathcal{C})$ respectively.

First we show that if $\mathcal{C}$ and $b(\mathcal{C})$ are ideal then, for all $\ell, w \in R^n_+$, $\alpha\beta \le w^T\ell$ where $\alpha \equiv \min\{w(C) : \ C \in E(\mathcal{C})\}$ and $\beta \equiv \min\{\ell(D) : D \in E(b(\mathcal{C}))\}$.

If $\alpha = 0$ or $\beta = 0$, then this clearly holds.

If $\alpha > 0$ and $\beta > 0$, we can assume w.l.o.g. that $\alpha = \beta = 1$ by scaling ℓ and w. So $Aw \ge \mathbf{1}$, i.e. w belongs to the polyhedron $P \equiv \{x \ge 0, \ Ax \ge \mathbf{1}\}$. Therefore w is greater than or equal to a convex combination of the extreme points of P, which are the rows of B by Remark 1.16(i) since P is an integral polyhedron. It follows that $w^T \ge \lambda^T B$ where $\lambda \ge 0$ and $\sum_i \lambda_i = 1$. Similarly, one shows that $\ell^T \ge \mu^T A$ where $\mu \ge 0$ and $\sum_i \mu_i = 1$. Since $BA^T \ge J$, where J denotes the matrix of all 1's, it follows that

$$w^T\ell \ge \lambda^T BA^T\mu \ge \lambda^T J\mu = 1 = \alpha\beta$$

Now we prove the converse. Let $\mathcal{C}$ be a nontrivial clutter and let w be any extreme point of $P \equiv \{x \ge 0 : \ Ax \ge \mathbf{1}\}$. Since $Aw \ge \mathbf{1}$, it follows that $\min\{w(C) : \ C \in E(\mathcal{C})\} \ge 1$. For any point z in $Q \equiv \{z \ge 0 : \ Bz \ge \mathbf{1}\}$, we also have $\min\{z(D) : \ D \in E(b(\mathcal{C}))\} \ge 1$. Using the hypothesis, it follows that $w^Tz \ge 1$ is satisfied by all points z in Q. Furthermore, equality holds for at least one $z \in Q$. Now, by linear programming duality,

$$1 \ = \ \min\{w^Tz : \ z \in Q\} \ = \ \max\{\mu^T\mathbf{1} : \ \mu^T B \le w^T, \ \mu \ge 0\}.$$

It follows from Remark 1.16(ii) that w is a 0,1 extreme point of P. Therefore, $\mathcal{C}$ is ideal. By Theorem 1.17, $b(\mathcal{C})$ is also ideal. □

1.3.2 Two-Commodity Flows

Let G be an undirected graph and let $\{s_1, t_1\}$ and $\{s_2, t_2\}$ be two pairs of nodes of G with $s_1 \ne t_1$ and $s_2 \ne t_2$. A *two-commodity cut* is a minimal set of edges separating each of the pairs $\{s_1, t_1\}$ and $\{s_2, t_2\}$. A *two-commodity path* is an s_1t_1-path or an s_2t_2-path.

For any edge capacities $w \in R^{E(G)}_+$, Hu [126] showed that a minimum capacity two-commodity cut can be obtained by solving the linear program (1.1), where M is the incidence matrix of two-commodity paths versus edges.

Theorem 1.22 (Hu [126]) *The clutter of two-commodity paths is ideal.*

This theorem states that the polyhedron

$$\begin{array}{rcll} x(P) & \geq & 1 & \text{for all two-commodity paths } P \\ x_e & \geq & 0 & \text{for all } e \in E(G) \end{array}$$

is integral. Using Lehman's theorem (Theorem 1.17), the clutter of two-commodity cuts is ideal.

The clutters of two-commodity paths and of two-commodity cuts do not pack, but both have the 1/2-MFMC property (Hu [126] and Seymour [184], respectively).

For more than two commodities, the clutter of multicommodity paths is not always ideal but conditions on the graph G and the source-sink pairs $\{s_1, t_1\}, \ldots, \{s_k, t_k\}$ have been obtained under which it is ideal. See Papernov [159], Okamura and Seymour [151], Lomonosov [135] and Frank [85] for examples.

1.3.3 r-Cuts and r-Arborescences

Consider a connected digraph (N, A) with $r \in N$ and nonnegative integer arc lengths ℓ_a for $a \in A$. An *r-arborescence* is a minimal arc set that contains an rv-dipath for every $v \in N$. It follows that an r-arborescence contains $|N| - 1$ arcs forming a spanning tree and each node of $N - \{r\}$ is entered by exactly one arc. The minimal transversals of the clutter of r-arborescences are called *r-cuts.*

Theorem 1.23 (Fulkerson [91]) *The clutter of r-cuts has the MFMC property, i.e. the minimum length of an r-arborescence is equal to the maximum number of r-cuts such that each $a \in A$ is contained in at most ℓ_a of them.*

In other words, both sides of the linear programming duality relation

$$\min \quad \{\textstyle\sum_{a \in A} \ell_a x_a : \quad x(C) \geq 1 \quad \text{for all } \ r\text{-cuts } C$$

$$
\begin{aligned}
&& x_a \geq 0\} \\
= \max \quad \{\textstyle\sum_{C\ r\text{-cut}} y_C : && \sum_{C \ni a} y_C \leq \ell_a \quad \text{for all} \quad a \in A \\
&& y_C \geq 0\}
\end{aligned}
$$

have integral optimal solutions x and y.

Theorem 1.24 (Edmonds [79]) *The clutter of r-arborescences has the MFMC property.*

In other words, both sides of the linear programming duality relation

$$
\begin{aligned}
\min \{\textstyle\sum_{a\in A} \ell_a x_a : && x(B) \geq 1 \quad \text{for all} \quad r\text{-arborescences } B \\
&& x_a \geq 0\} \\
= \max \quad \{\textstyle\sum_{B\ r\text{-arborescence}} y_B : && \sum_{B \ni a} y_B \leq \ell_a \quad \text{for all} \quad a \in A \\
&& y_B \geq 0\}
\end{aligned}
$$

have integral optimal solutions x and y. The fact that the minimization problem has an integral optimal solution x follows from Theorems 1.17 and 1.23, but the fact that the dual also does cannot be deduced from these theorems.

1.3.4 Dicuts and Dijoins

Let (N, A) be a digraph. An arc set $C \subseteq A$ is called a *dicut* if there exists a nonempty node set $S \subset N$ such that $(S, N - S) = C$ and $(N - S, S) = \emptyset$ where (S_1, S_2) denotes the set of arcs ij with $i \in S_1$ and $j \in S_2$, and C is minimal with this property. A *dijoin* is a minimal arc set intersecting every dicut.

Theorem 1.25 (Lucchesi-Younger [141]) *The clutter of dicuts has the MFMC property.*

By Lehman's theorem, it follows that the clutter of dijoins is ideal. However, Schrijver [171] showed by an example that the clutter of dijoins does not always have the MFMC property. Two additional examples are given in [63].

Conjecture 1.26 (Woodall [210]) *The clutter of dijoins packs.*

1.3.5 T-Cuts and T-Joins

Let G be an undirected graph and $T \subseteq V(G)$ a node set of even cardinality. Such a pair (G, T) is called a *graft*. An edge set $J \subseteq E(G)$ is a *T-join* if it induces an acyclic graph where the odd nodes coincide with T. The minimal transversals of the clutter of T-joins are called *T-cuts*. For disjoint node sets S_1, S_2, let (S_1, S_2) denote the set of edges with one endnode in S_1 and the other in S_2. T-cuts are edge sets of the form $(S, V(G) - S)$ where $|T \cap S|$ is odd.

When $T = \{s, t\}$, the T-joins are the *st*-paths of G and the T-cuts are the *st*-cuts.

When $T = V(G)$, the T-joins of size $|V(G)|/2$ are the perfect matchings of G.

Theorem 1.27 (Edmonds-Johnson [82]) *The clutter of T-cuts is ideal.*

Hence, the polyhedron

$$\begin{aligned} x(C) &\geq 1 \quad \text{for all } T\text{-cuts } C \\ x_e &\geq 0 \quad \text{for all } e \in E(G) \end{aligned}$$

is integral.

The Edmonds-Johnson theorem together with Lehman's theorem (Theorem 1.17) implies that the clutter of T-joins is also ideal. Hence the polyhedron

$$\begin{aligned} x(J) &\geq 1 \quad \text{for all } T\text{-joins } J \\ x_e &\geq 0 \quad \text{for all } e \in E(G) \end{aligned}$$

is integral.

The clutter of T-cuts does not pack, but it has the 1/2-MFMC property (Seymour [188]). The clutter of T-joins does not have the 1/2-MFMC property (there is an example requiring multiplication by 4 to get an integer dual), but it may have the 1/4-MFMC property (open problem). Seymour [185] showed that the 1/4-MFMC property would follow from a conjecture of Fulkerson, namely Conjecture 1.32 mentioned later in this chapter. Another intriguing conjecture is the following. In a graph G, a *postman set* is a T-join where T coincides with the odd degree nodes of G.

Conjecture 1.28 (Conforti and Johnson [55]) *The clutter of postman sets packs in graphs not contractible to the Petersen graph.*

If true, this implies the four color theorem (see Exercise 1.31)!
We discuss T-cuts and T-joins in greater detail in Chapter 2.

1.3.6 Odd Cycles in Graphs

Let G be an undirected graph and $\mathcal{C}$ the clutter of odd cycles, i.e. $V(\mathcal{C}) = E(G)$ and $E(\mathcal{C})$ is the family of odd cycles in G (viewed as edge sets). Seymour [183] characterized exactly when the clutter of odd cycles has the MFMC property and Guenin [110] characterized exactly when it is ideal. These results are described in Chapter 5 in the more general context of *signed graphs*.

1.3.7 Edge Coloring of Graphs

In a simple graph G, consider the clutter $\mathcal{C}$ whose vertices are the maximal matchings of G and whose edges are indexed by the edges of G, with $S_e \in E(\mathcal{C})$ being the set of maximal matchings that contain edge e. The problem

$$\chi^{'}(G) = \min\{\mathbf{1}x : \ M(\mathcal{C})x \geq \mathbf{1}, \ x \geq 0 \ integral\} \tag{1.7}$$

is that of finding the edge-chromatic number of G. The dual problem

$$\max\{y\mathbf{1} : \ yM(\mathcal{C}) \leq \mathbf{1}, \ y \geq 0 \ integral\}$$

is that of finding a maximum number of edges in G no two of which belong to the same matching. This equals the maximum degree $\Delta(G)$, except in the trivial case where $\Delta(G) = 2$ and G contains a triangle. By Vizing's theorem [205], $\chi^{'}(G)$ and $\Delta(G)$ differ by at most one. It is NP-complete to decide whether $\chi^{'}(G) = \Delta(G)$ or $\Delta(G) + 1$ (Holyer [123]). For the Petersen graph, $\chi^{'}(G) = 4 > \Delta(G) = 3$. The following conjecture of Tutte [203] was proved recently.

Theorem 1.29 (Robertson, Sanders, Seymour, Thomas [165], [169], [170]) *Every 2-connected cubic graph that is not contractible to the Petersen graph is 3-edge-colorable.*

The Petersen graph is nonplanar. So, by specializing the above theorem to planar graphs, we get the following corollary.

Theorem 1.30 (Appel and Haken [2]. See also [164]) *Every 2-connected cubic planar graph is 3-edge-colorable.*

This is equivalent to the famous 4-color theorem for planar maps, as shown by Tait [190] over a century ago. In fact, Tait mistakenly believed that he had settled the 4-color conjecture because he thought that every 2-connected cubic planar graph is Hamiltonian (which would imply Theorem 1.30). Tutte [200] found a counterexample over sixty years later!

Exercise 1.31 *Show that Conjecture 1.28 implies Theorem 1.29 and therefore the* 4*-color theorem.*

$$\text{Let} \quad \chi_2'(G) = min\{\mathbf{1}x : \ M(\mathcal{C})x \geq \mathbf{1}, \ x \geq 0, \ 2x \ integral\}. \qquad (1.8)$$

For the Petersen graph, it is easy to check that $\chi_2'(G) = 3$.

Conjecture 1.32 (Fulkerson [89]) *For every 2-connected cubic graph,* $\chi_2'(G) = 3$.

Seymour [185] showed that Fulkerson's conjecture holds if one relaxes the condition $x \geq 0$ in (1.8).

1.3.8 Feedback Vertex Set

Given a digraph $D = (V, A)$, a vertex set $S \subseteq V$ is called a *feedback vertex set* if $V(C) \cap S \neq \emptyset$ for every directed cycle C. Let $\mathcal{C}$ denote the clutter with $V(\mathcal{C}) = V$ and $E(\mathcal{C})$ the family of minimal directed cycles (viewed as sets of vertices). Then a feedback vertex set is a transversal of $\mathcal{C}$. Guenin and Thomas [113] characterize the digraphs D for which $\mathcal{C}$ packs for every subdigraph H of D. Cai, Deng and Zang [23] and [24] consider the feedback vertex set problem in tournaments and bipartite tournaments respectively. A *tournament* is an orientation of a complete graph. A *bipartite tournament* is an orientation of a complete bipartite graph. In the first case, $\mathcal{C}$ consists of the directed triangles in D and, in the second case, $\mathcal{C}$ consists of the directed squares (check this). Cai, Deng and Zang [23], [24] characterize the tournaments and bipartite tournaments D for which $\mathcal{C}$ has the MFMC property.

Recently, Ding and Zang [77] solved a similar problem on undirected graphs. They characterized in terms of forbidden subgraphs the graphs G for which the clutter $\mathcal{C}$ of cycles has the MFMC property. Here $V(\mathcal{C}) \equiv V(G)$ and $E(\mathcal{C})$ is the family of cycles of G viewed as sets of vertices.

1.4 Deletion, Contraction and Minor

Let $\mathcal{C}$ be a clutter. For $j \in V(\mathcal{C})$, the *contraction* $\mathcal{C}/j$ and *deletion* $\mathcal{C} \setminus j$ are clutters defined as follows: both have $V(\mathcal{C}) - \{j\}$ as vertex set, $E(\mathcal{C}/j)$ is the set of minimal members of $\{S - \{j\} : \; S \in E(\mathcal{C})\}$ and $E(\mathcal{C} \setminus j) = \{S \in E(\mathcal{C}) : \; j \notin S\}$.

Exercise 1.33 *Given an undirected graph G, consider the clutter $\mathcal{C}$ whose vertices are the edges of G and whose edges are the cycles of G (viewed as edge sets). Describe $\mathcal{C}\backslash j$ and $\mathcal{C}/j$. Relate to the graph-theoretic notions of edge deletion and edge contraction in G.*

Contractions and deletions of distinct vertices of $\mathcal{C}$ can be performed sequentially, and it is easy to show that the result does not depend on the order.

Proposition 1.34 *For a clutter $\mathcal{C}$ and distinct vertices j_1, j_2,*

(i) $(\mathcal{C}\backslash j_1)\backslash j_2 = (\mathcal{C}\backslash j_2)\backslash j_1$

(ii) $(\mathcal{C}/j_1)/j_2 = (\mathcal{C}/j_2)/j_1$

(iii) $(\mathcal{C}\backslash j_1)/j_2 = (\mathcal{C}/j_2)\backslash j_1$

Proof: Use the definitions of contraction and deletion! $\square$

Contracting $j \in V(\mathcal{C})$ corresponds to setting $x_j = 0$ in the set covering constraints $Mx \geq \mathbf{1}$ of (1.1) since column j is removed from M as well as the resulting dominating rows. Deleting j corresponds to setting $x_j = 1$ since column j is removed from M as well as all the rows with a 1 in column j.

A clutter $\mathcal{D}$ obtained from $\mathcal{C}$ by a sequence of deletions and contractions is a *minor* of $\mathcal{C}$. For disjoint subsets V_1 and V_2 of $V(\mathcal{C})$, we let $\mathcal{C}/V_1\backslash V_2$ denote the minor obtained from $\mathcal{C}$ by contracting all vertices in V_1 and deleting all vertices in V_2. If $V_1 \neq \emptyset$ or $V_2 \neq \emptyset$, the minor is *proper*.

Proposition 1.35 *For a clutter $\mathcal{C}$ and $U \subset V(\mathcal{C})$,*

(i) $b(\mathcal{C}\backslash U) = b(\mathcal{C})/U$

(ii) $b(\mathcal{C}/U) = b(\mathcal{C})\backslash U$

Proof: Use the definitions of contraction, deletion and blocker! □

Proposition 1.36 (Seymour [183]) *If a clutter $\mathcal{C}$ has the MFMC property, then so do all its minors.*

Proof: Trivial clutters have the MFMC property. So let $\mathcal{C}' = \mathcal{C}/V_1\backslash V_2$ be a nontrivial minor of $\mathcal{C}$. It suffices to show that, for every $w' \in Z_+^{V(\mathcal{C}')}$,

$$\max\{y\mathbf{1} : y \geq 0,\ yM(\mathcal{C}') \leq w'\} \tag{1.9}$$

has an integral optimal solution. Since $b(\mathcal{C}')$ is nontrivial (Exercise 1.12), $\tau = \min\{w'(B') : B' \in E(b(\mathcal{C}'))\}$ is well defined. Define w by

$$\begin{aligned} w_j &= w'_j \quad for\ j \in V(\mathcal{C}') \\ w_j &= \tau \quad for\ j \in V_1 \\ w_j &= 0 \quad for\ j \in V_2. \end{aligned}$$

If B is an edge of $b(\mathcal{C})$ and $B \cap V_1 \neq \emptyset$, then $w(B) \geq \tau$. On the other hand, if $B \cap V_1 = \emptyset$, then, by Proposition 1.35, $b(\mathcal{C}') = b(\mathcal{C}) \setminus V_1/V_2$ and therefore B contains an edge B' of $b(\mathcal{C}')$ and $w(B) \geq w'(B') \geq \tau$.

Furthermore, there exists B with $w(B) = \tau$. Since $\mathcal{C}$ has the MFMC property, it follows that

$$\max\{y\mathbf{1} : y \geq 0,\ yM(\mathcal{C}) \leq w\}$$

has an integral optimal solution y^* with function value τ. y^* can be used to construct a solution y^{**} of (1.9) with function value τ as follows. Start with $y^{**} = 0$. Let A be an edge of $\mathcal{C}$ such that $y^*_A > 0$. The fact that $w_j = 0$ for $j \in V_2$ implies that $A \cap V_2 = \emptyset$. Hence A contains an edge A' of $\mathcal{C}'$. Increase $y^{**}_{A'}$ by y^*_A. Repeat for each A such that $y^*_A > 0$. □

Similarly, one may prove the following result.

Proposition 1.37 *If a clutter is ideal, then so are all its minors.*

Exercise 1.38 *Prove Proposition 1.37.*

Corollary 1.39 *Let M be a 0,1 matrix. The following are equivalent.*

- *The polyhedron $\{x \geq 0 :\ Mx \geq \mathbf{1}\}$ is integral.*
- *The polytope $\{0 \leq x \leq \mathbf{1} :\ Mx \geq \mathbf{1}\}$ is integral.*

Propositions 1.36 and 1.37 suggest the following concepts.

Definition 1.40 *A clutter is* minimally non MFMC *if it does not have the MFMC property but all its proper minors do.*

A clutter is minimally nonideal *if it is not ideal but all its proper minors are.*

A clutter is minimally nonpacking *if it does not pack but all its proper minors do.*

Properties of these clutters are investigated in Chapters 4 and 5.

Chapter 2

T-Cuts and T-Joins

Consider a connected graph G with nonnegative edge weights w_e, for $e \in E(G)$. The Chinese Postman Problem consists in finding a minimum weight closed walk going through each edge at least once (the edges of the graph represent streets where mail must be delivered and w_e is the length of the street). Equivalently, the postman must find a minimum weight set of edges $J \subseteq E(G)$ such that $J \cup E(G)$ induces an Eulerian graph, i.e. J induces a graph the odd degree nodes of which coincide with the odd degree nodes of G. Since $w \geq 0$, we can assume w.l.o.g. that J is acyclic. Such an edge set J is called a *postman set*.

The problem is generalized as follows. Let G be a graph and T a node set of G of even cardinality. An edge set J of G is called a *T-join* if it induces an acyclic graph the odd degree nodes of which coincide with T. For disjoint node sets S_1, S_2, let (S_1, S_2) denote the set of edges with one endnode in S_1 and the other in S_2. A *T-cut* is a minimal edge set of the form $(S, V(G) - S)$ where S is a set of nodes with $|T \cap S|$ odd. Clearly every T-cut intersects every T-join.

Edmonds and Johnson [82] considered the problem of finding a minimum weight T-join. One way to solve this problem is to reduce it to the perfect matching problem in a complete graph K_p, where $p = |T|$. Namely, compute the lengths of shortest paths in G between all pairs of nodes in T, use these values as edge weights in K_p and find a minimum weight perfect matching in K_p. The union of the corresponding paths in G is a minimum weight T-join. There is another way to solve the minimum weight T-join problem: Edmonds and Johnson gave a direct primal-dual algorithm and, as a by-product, obtained that the clutter of T-cuts is ideal.

Theorem 2.1 (Edmonds-Johnson [82])
The polyhedron

$$\begin{aligned} x(C) &\geq 1 \quad \textit{for all} \;\; T-\textit{cuts} \;\; C && (2.1)\\ x_e &\geq 0 \quad \textit{for all} \quad e \in E(G). && (2.2)\end{aligned}$$

is integral.

In the next section, we give a non-algorithmic proof of this theorem suggested by Pulleyblank [162].

The clutter of T-cuts does not pack, but Seymour [188] showed that it has the 1/2-MFMC property. In Section 2.2, we prove Seymour's result, following a short argument of Sebö [174] and Conforti [39].

As we have seen in Chapter 1, clutters come in pairs: To each clutter $\mathcal{C}$, we can associate its blocker $b(\mathcal{C})$ whose edges are the minimal transversals of $\mathcal{C}$. Lehman [132] showed that $\mathcal{C}$ is ideal if and only if $b(\mathcal{C})$ is ideal (Theorem 1.17). The Edmonds-Johnson theorem together with Lehman's theorem implies that the clutter of T-joins is also ideal, i.e. the polyhedron

$$\begin{aligned} x(J) &\geq 1 \quad \text{for all} \;\; T-\text{joins} \;\; J\\ x_e &\geq 0 \quad \text{for all} \quad e \in E(G).\end{aligned}$$

is integral. The clutter of T-joins does not pack in general. In Section 2.3, we present two special cases where it does.

2.1 Proof of the Edmonds-Johnson Theorem

First, we prove the following lemma. For $v \in V(G)$, let $\delta(v)$ denote the set of edges incident with v. A *star* is a tree where one node is adjacent to all the other nodes.

Lemma 2.2 *Let $\tilde{x}$ be an extreme point of the polyhedron*

$$\begin{aligned} x(\delta(v)) &\geq 1 \quad \textit{for all} \quad v \in T & (2.3)\\ x_e &\geq 0 \quad \textit{for all} \quad e \in E(G). & (2.4) \end{aligned}$$

The connected components of the graph $\tilde{G}$ induced by the edges such that $\tilde{x}_e > 0$ are either

(i) *odd cycles with nodes in T and edges $\tilde{x}_e = 1/2$, or*

(ii) *stars with nodes in T, except possibly the center, and edges $\tilde{x}_e = 1$.*

Proof: Every connected component C of $\tilde{G}$ is either a tree or contains a unique cycle, since the number of edges in C is at most the number of inequalities (2.3) that hold with equality.

Assume first that C contains a unique cycle. Then (2.3) holds with equality for all nodes of C, which are therefore in T. Now C is a cycle since, otherwise, C has a pendant edge e with $\tilde{x}_e = 1$ and therefore C is disconnected, a contradiction. If C is an even cycle, then by alternately increasing and decreasing $\tilde{x}$ around the cycle by a small ϵ ($-\epsilon$ respectively), $\tilde{x}$ can be written as a convex combination of two points satisfying (2.3) and (2.4). So (i) must hold.

Assume now that C is a tree. Then (2.3) holds with equality for at least $|V(C)| - 1$ nodes of C. In particular, it holds with equality for at least one node of degree one. Since C is connected, this implies that C is a star and (ii) holds. $\square$

Proof Theorem 2.1: In order to prove the theorem, it suffices to show that every extreme point $\tilde{x}$ of the polyhedron (2.1)–(2.2) is the incidence vector of a T-join. We proceed by induction on the number of nodes of G.

Suppose first that $\tilde{x}$ is an extreme point of the polyhedron (2.3)–(2.4). Consider a connected component of the graph $\tilde{G}$ induced by the edges such that $\tilde{x}_e > 0$ and let S be its node set. Since $\tilde{x}(S, V(G) - S) = 0$, it follows from (2.1) that S contains an even number of nodes of T. By Lemma 2.2, $\tilde{G}$ contains no odd cycle, showing that $\tilde{x}$ is an integral vector. Furthermore, $\tilde{x}$ is the incidence vector of a T-join since, by Lemma 2.2 again, the component of $\tilde{G}$ induced by S is a star and

$|S \cap T|$ even implies that the center is in T if and only if the star has an odd number of edges.

Assume now that $\tilde{x}$ is not an extreme point of the polyhedron (2.3)–(2.4). Then there is some T-cut $C = (V_1, V_2)$ with $|V_1| \geq 2$ and $|V_2| \geq 2$ such that

$$\tilde{x}(C) = 1.$$

Let $G_1 = (V_1 \cup \{v_2\}, E_1)$ be the graph obtained from G by contracting V_2 to a single node v_2. Similarly, $G_2 = (V_2 \cup \{v_1\}, E_2)$ is the graph obtained from G by contracting V_1 to a single node v_1. The new nodes v_1, v_2 belong to T. For $i = 1, 2$, let $\tilde{x}^i$ be the restriction of $\tilde{x}$ to E_i. Since every T-cut of G_i is also a T-cut of G, it follows by induction that $\tilde{x}^i$ is greater than or equal to a convex combination of incidence vectors of T-joins of G_i. Let $\mathcal{T}_i$ be this set of T-joins. Each T-join in $\mathcal{T}_i$ has exactly one edge incident with v_i. Since $\tilde{x}^1$ and $\tilde{x}^2$ coincide on the edges of C, it follows that the T-joins of $\mathcal{T}_1$ can be combined with those of $\mathcal{T}_2$ to form T-joins of G and that $\tilde{x}$ is greater than or equal to a convex combination of incidence vectors of T-joins of G. Since $\tilde{x}$ is an extreme point, it is the incidence vector of a T-join. $\Box$

We have just proved that the clutter of T-cuts is ideal. It does not have the MFMC property in general graphs. However Seymour proved that it does in bipartite graphs. Seymour also showed that, in a general graph, if the edge weights w_e are integral and their sum is even in every cycle, then the dual variables can be chosen to be integral in an optimum solution. We prove these results in the next section.

2.2 Packing T-Cuts

2.2.1 Theorems of Seymour and Lovász

The purpose of this section is to prove the following theorems.

Theorem 2.3 (Seymour [188]) *In a bipartite graph, the clutter of T-cuts packs, i.e. the minimum cardinality of a T-join equals the maximum number of disjoint T-cuts.*

Theorem 2.4 (Lovász [138]) *In a graph, the clutter of T-cuts has the 1/2-MFMC property.*

Exercise 2.5 *Consider the complete graph K_4 on four nodes and let $|T| = 4$. Show that there are exactly seven T-joins. Describe the T-cuts. Do you see any relation with Q_6 (see Exercise 1.18 for the definition of Q_6)?*

We give a proof of Theorem 2.3 based on ideas of Sebö [174] and Conforti [39].

Given a graph G and a T-join J, let G_J be the weighted graph obtained by assigning weights -1 to the edges of J and $+1$ to all the other edges.

Remark 2.6

(i) J is a minimum T-join if and only if G_J has no negative cycle.

(ii) If J is a minimum T-join and C is a 0-weight cycle in G_J, then $J \Delta C$ is a minimum T-join.

Proof of Theorem 2.3: The result is trivial for $|T| = 0$, so assume $|T| \geq 2$. The proof is by induction on the number of nodes of the bipartite graph G. Let J be a minimum T-join chosen so that its longest path $Q \subseteq J$ is longest possible among all minimum T-joins. Since J is acyclic, the endnodes of Q have degree 1 in J, so both are in T. Let u be an endnode of Q and let x be the neighbor of u in Q. Since J is minimum, G_J has no negative cycle. We claim that every 0-weight cycle of G_J that contains node u also contains edge ux. Suppose otherwise. If C contains some other node of Q, then $Q \cup C$ contains a negative cycle (check this!), a contradiction to J being a minimum T-join. If u is the unique node of Q in C, then $J \Delta C$ is a minimum T-join (by Remark 2.6) with a longer path than Q, a contradiction to our choice of J. So the claim holds and, since G is bipartite,

(*) every cycle that contains node u but not edge ux has weight at least 2 in G_J.

Let U be the node set comprising u and its neighbors in G. Let G^* be the bipartite graph obtained from G by contracting U into a single node u^*. If $|U \cap T|$ is even, set $T^* = T \setminus U$ and if $|U \cap T|$, set

$T^* = (T \setminus U) \cup \{u^*\}$. Let $J^* = J \cap E(G^*)$. Then J^* is a T^*-join of G^* and, by (*) the graph $G^*_{J^*}$ has no negative cycle. So J^* is a minimum T^*-join of G^* by Remark 2.6.

Now, by induction, G^* has $|J^*|$ disjoint T^*-cuts. Since $\delta(v)$ is a T-cut of G disjoint from them, G has $|J^*| + 1 = |J|$ disjoint T-cuts. Since G can have at most $|J|$ disjoint T-cuts, the theorem holds. □

This result implies the following theorem of Lovász [138].

Theorem 2.7 *In a graph G, the minimum cardinality of a T-join is equal to one half of the maximum cardinality of a set of T-cuts such that no edge belongs to more than two T-cuts in the set.*

Proof: Subdivide each edge of G by a new node and apply Theorem 2.3. □

Exercise 2.8 *Prove Theorem 2.4 using Theorem 2.7.*

Another useful consequence of Theorem 2.3 is the following result of Seymour [188].

Definition 2.9 *Let G be a graph and let $w \in Z_+^{E(G)}$. The* even cycle property *holds if, in every cycle of G, the sum of the weights is even.*

Theorem 2.10 (Seymour [188]) *Assume graph G and weight vector $w \in Z_+^{E(G)}$ satisfy the even cycle property. Then the minimum weight of a T-join equals the maximum number of T-cuts such that no edge e belongs to more than w_e of these T-cuts.*

Proof: If $w_e > 0$, subdivide edge e into w_e edges, each of weight 1. If $w_e = 0$, contract edge e and consider the resulting node as being in T if exactly one of the endnodes of e in G belongs to T. Now the theorem follows from Theorem 2.3. □

The next result follows from a difficult theorem of Seymour on binary clutters with the MFMC property (Theorem 5.30). A graph G can be *T-contracted to K_4* if its node set V can be partitioned into $\{V_1, V_2, V_3, V_4\}$ so that each V_i induces a connected graph containing an odd number of nodes of T and, for each $i \neq j$, there is an edge with endnodes in V_i and V_j.

Theorem 2.11 *In a graph that cannot be T-contracted to K_4, the clutter of T-cuts has the MFMC property.*

2.2.2 More Min Max Results

Theorem 2.4 can be strengthened as follows. For any node set X, denote by $q_T(X)$ the number of connected components of $G \setminus X$ that contain an odd number of nodes of T.

Theorem 2.12 (Frank, Sebö, Tardos [86]) *In a graph G, the minimum cardinality of a T-join is equal to $\frac{1}{2} max\{\sum_i q_T(V_i)\}$, where the maximum is taken over all partitions $\{V_1, \ldots, V_\ell\}$ of V.*

This theorem can be used to prove Tutte's theorem on perfect matchings.

Theorem 2.13 (Tutte [201]) *A graph contains no perfect matching if and only if there exists a node set X such that $G \setminus X$ contains at least $|X| + 1$ components of odd cardinality.*

Proof: (Frank and Szigeti [87]) Apply Theorem 2.12 with the choice $T = V$. Note that in this case, $q_T(X)$ is the number of components of odd cardinality in $G \setminus X$. If there is no perfect matching, the minimum cardinality of a T-join is larger than $\frac{1}{2}|V|$. By Theorem 2.12, there is a partition $\{V_1, \ldots, V_\ell\}$ of V such that $\frac{1}{2} \sum_i q_T(V_i) > \frac{1}{2}|V|$. Therefore, there must be a subscript i such that $q_T(V_i) > |V_i|$, that is, the number of components of $G \setminus V_i$ with odd cardinality is larger than $|V_i|$, as required. $\Box$

Sebö proved yet another min-max theorem concerning T-joins. A *multicut* is an edge set whose removal disconnects G into two or more connected components. If each of these connected components contains an odd number of nodes of T, the multicut is called a *T-border*. Clearly, the number k of connected components in a T-border is even. The *value* of the T-border B is defined to be $val(B) = \frac{k}{2}$.

Theorem 2.14 (Sebö [175]) *In a graph G, the minimum cardinality of a T-join is equal to $max\{\sum_i val(B_i)\}$, where the maximum is taken over all edge disjoint borders $\{B_1, \ldots, B_\ell\}$.*

2.3 Packing T-Joins

The following conjecture is open.

Conjecture 2.15 *The clutter of T-joins has the* $1/4$*-MFMC property.*

By contrast, characterizing when the clutter of T-joins has the MFMC property is settled. This is because T-joins form a binary clutter. So the answer follows from Seymour's characterization [183] of the binary clutters with the MFMC property (Theorem 5.30).

A *graft* (G, T) is a graph together with a node set T of even cardinality. A *minor* of a graft is obtained by performing a sequence of edge deletions and edge contractions where, in contracting edge uv into node w, we put w in T if exactly one of the nodes u, v was in T. An *odd-$K_{2,3}$* is the graft consisting of the complete bipartite graph $K_{2,3}$ and a set T of four nodes containing all three nodes of degree two and exactly one of the nodes of degree three. Seymour's theorem [183] implies the following.

Theorem 2.16 *In a graft without odd-$K_{2,3}$ minor, the clutter of T-joins has the MFMC-property.*

Codato, Conforti and Serafini [36] gave a direct graphical proof of this result.

Conforti and Johnson [55] conjectured that the clutter of postman sets packs in graphs noncontractible to the Petersen graph (Conjecture 1.28). They were able to show the following result. A 4-*wheel* W_4 is a graph on five nodes where four of the nodes induce a hole H and the fifth node is adjacent to all the nodes of H. A graph contractible to a 4-wheel is said to have a *4-wheel minor*. Graphs without 4-wheel minors are planar, by Kuratowski's theorem and the observation that K_5 and $K_{3,3}$ are contractible to a 4-wheel.

Theorem 2.17 (Conforti and Johnson [55]) *In a graph without* 4*-wheel minors, the clutter of postman sets has the MFMC property.*

Chapter 3

Perfect Graphs and Matrices

Chapter 1 discussed the min-max equation (1.1)=(1.2). In this chapter, we consider the max-min equation

$$\begin{aligned} &\max\{wx : x \geq 0,\ Mx \leq \mathbf{1}\} \\ &= \min\{y\mathbf{1} : y \geq 0,\ yM \geq w\}. \end{aligned}$$

A $0,1$ matrix M with no column of zeroes is *perfect* if the polytope $P = \{x \geq 0 : \ Mx \leq \mathbf{1}\}$ is integral, i.e. all the extreme points of P are 0,1 vectors. When M is perfect, the linear program $\max\{wx : x \geq 0,\ Mx \leq \mathbf{1}\}$ has an integral optimal solution x for all $w \in R^n$. Therefore, the *set packing problem* $\max\{wx : \ Mx \leq \mathbf{1},\ x \in \{0,1\}^n\}$ is solvable in polynomial time. By contrast, for a general 0,1 matrix M, the set packing problem is NP-hard [94].

Edmonds and Giles [81] observed that, when a linear system $Ax \leq b,\ x \geq 0$ is TDI and b is integral, the polyhedron $\{x : \ Ax \leq b,\ x \geq 0\}$ is integral. The converse is not true in general. But it is true when A is a 0,1 matrix and $b = \mathbf{1}$, as shown by Lovász.

Theorem 3.1 (Lovász [136]) *For a 0,1 matrix M with no column of zeroes, the following statements are equivalent:*

(i) the linear system $Mx \leq \mathbf{1},\ x \geq 0$ is TDI,

(ii) the matrix M is perfect,

(iii) max $\{wx : Mx \le \mathbf{1},\ x \ge 0\}$ *has an integral optimal solution* x *for all* $w \in \{0,1\}^n$.

Clearly (i) implies (ii) implies (iii), where the first implication is the Edmonds-Giles property and the second follows from the definition of perfection. What is surprising is that (iii) implies (i) and, in fact, that (ii) implies (i). In this chapter, we prove Lovász's theorem. The proof uses a combination of graph theoretic and polyhedral arguments.

A graph is *perfect* if, in every node induced subgraph, the chromatic number equals the size of a largest clique. The concept of perfection is due to Berge [5]. The *clique-node matrix* of a graph is a 0,1 matrix M in which entry m_{ij} is 1 if and only if node j belongs to maximal clique i. Chvátal [30] established the following connection between perfect graphs and perfect matrices: A 0,1 clutter matrix with no column of zeroes is perfect if and only if it is the clique-node matrix of a perfect graph. A major open question is to characterize the graphs that are not perfect but all their proper node induced subgraphs are. These graphs are called *minimally imperfect*. A *hole* is a chordless cycle of length greater than three and it is *odd* if it contains an odd number of edges. Berge's [5] *strong perfect graph conjecture* states that odd holes and their complements are the only minimally imperfect graphs. This conjecture, made in 1960, is still open but it is known that minimally imperfect graphs are partitionable (see Section 3.4) and numerous properties of partitionable graphs are known.

3.1 The Perfect Graph Theorem

In this chapter, all graphs are simple. In a graph G, a *clique* is a set of pairwise adjacent nodes. The *clique number* $\omega(G)$ is the size of a largest clique in G. The *chromatic number* $\chi(G)$ is the smallest number of colors for the nodes so that adjacent nodes have distinct colors. Clearly, $\chi(G) \ge \omega(G)$, since every node of a clique has a different color.

Definition 3.2 *A graph G is* perfect *if* $\chi(G') = \omega(G')$ *for every node induced subgraph* G' *of* G.

Berge [5] conjectured and Lovász [136] proved that, if a graph G is perfect, then the complement $\bar{G}$ is also perfect. (The *complement* $\bar{G}$ is the graph having the same node set as G and having an edge between nodes i and j if and only if G does not.) This result is known as the *perfect graph theorem*. Lovász's proof, which we give in this section, is polyhedral. In Section 3.4 we give a nonpolyhedral proof due to Gasparyan [95].

Lemma 3.3 (The Replication Lemma) *Let G be a perfect graph and $v \in V(G)$. Create a new node v' and join it to v and to all the neighbors of v. Then the resulting graph G' is perfect.*

Proof: It suffices to show $\chi(G') = \omega(G')$ since, for induced subgraphs, the proof follows similarly. We distinguish two cases.

Case 1: Suppose v is contained in some maximum clique of G. Then $\omega(G') = \omega(G) + 1$. This implies $\chi(G') \leq \omega(G')$, since at most one new color is needed in G'. Clearly $\chi(G') = \omega(G')$ follows.

Case 2: Now suppose v is not contained in any maximum clique of G. Consider any coloring of G with $\omega(G)$ colors and let A be the color class containing v. Then, $\omega(G \setminus (A - \{v\})) = \omega(G) - 1$, since every maximum clique in G meets $A - \{v\}$. By the perfection of G, the graph $G \setminus (A - \{v\})$ can be colored with $\omega(G) - 1$ colors. Using one additional color for the nodes $(A - \{v\}) \cup \{v'\}$, we obtain a coloring of G' with $\omega(G)$ colors. $\square$

The next theorem includes results of Fulkerson [89], Lovász [136] and Chvátal [30]. A *stable set* of G is a set of pairwise nonadjacent nodes. The *stability number* $\alpha(G)$ is the size of a largest stable set in G.

Theorem 3.4 *For a graph G, the following are equivalent.*

(i) *G is perfect;*

(ii) *the polytope $P(G) = \{x \in R_+^{V(G)} : x(K) \leq 1 \;\; for\ all\ cliques\ K\}$ is integral, i.e. its extreme points are exactly the incidence vectors of the stable sets in G;*

(iii) *$\bar{G}$ is perfect.*

Proof: It suffices to show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), since $\overline{\overline{G}} = G$ gives (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Let $x \in P(G)$ be a rational vector. To prove (ii) we show that x is a convex combination of incidence vectors of stable sets in G. There exists a positive integer N such that $y = Nx$ is an integral vector. Let Y_i be disjoint sets such that $|Y_i| = y_i$, for $i \in V(G)$. Construct the graph G' with node set $\cup_{i \in V(G)} Y_i$ by joining every node of Y_i to every node of Y_j whenever $i = j$ or $ij \in E(G)$. This graph G' arises from G by deleting and replicating nodes repeatedly and hence, by Lemma 3.3, G' is perfect.

Let K' be a maximum clique in G' and let $K = \{i \in V(G) : K' \cap Y_i \neq \emptyset\}$. Then K is a clique and

$$\omega(G') = |K'| \leq \sum_{i \in K} |Y_i| \ = \ y(K) \ = \ Nx(K) \ \leq N \tag{3.1}$$

where the last inequality follows from $x \in P(G)$. Therefore G' can be colored with colors $1, \ldots, N$. Let $A_t = \{i \in V(G) : Y_i$ has a node with color $t\}$. Clearly, A_t is a stable set of G. Let x^{A_t} be its incidence vector. Now $y = \sum_{t=1}^N x^{A_t}$ follows by noting that each $i \in V(G)$ occurs in exactly $|Y_i| = y_i$ of the sets A_t, since Y_i induces a clique and hence its nodes have different colors. Therefore $x = \frac{1}{N} \sum_{t=1}^N x^{A_t}$.

(ii) $\Rightarrow$ (iii): It is easy to see that property (ii) is inherited by induced subgraphs, and therefore it suffices to show that $\chi(\bar{G}) = \omega(\bar{G})$. The proof is by induction on $|V(G)|$. Consider the face F of $P(G)$ defined by the hyperplane $x(V(G)) = \alpha(G)$. There is a facet of $P(G)$ of the form $x(K) \leq 1$ containing F, where K is a clique. Since K meets all the maximum stable sets of G, it follows that $\alpha(G \setminus K) = \alpha(G) - 1$, or equivalently $\omega(\bar{G} \setminus K) = \omega(\bar{G}) - 1$. By the induction hypothesis, $\bar{G} \setminus K$ can be colored with $\omega(\bar{G}) - 1$ colors. Using a new color for the nodes in K, we obtain an $\omega(\bar{G})$-coloring of $\bar{G}$. $\Box$

Exercise 3.5 *Let G be a perfect graph. Use the replication lemma to show that the linear system*

$$\begin{aligned} x(S) &\leq 1 \quad \textit{for all stable sets } S \\ x &\geq 0 \end{aligned}$$

is totally dual integral.

Exercise 3.6 *Let G be a perfect graph. Use the perfect graph theorem and the previous exercise to show that the linear system*

$$\begin{aligned} x(K) &\leq 1 \quad \textit{for all cliques } K \\ x &\geq 0 \end{aligned}$$

is totally dual integral.

3.2 Perfect Matrices

Definition 3.7 *A $0,1$ matrix M with no column of zeroes is* perfect *if the polytope $P(M) = \{x \in R^n_+ : Mx \leq \mathbf{1}\}$ has only integral vertices.*

Clearly, dominated rows do not affect this definition, so we assume in this section that M is a 0,1 clutter matrix. Chvátal [30] showed that M is a perfect matrix if and only if it is the clique-node matrix of a perfect graph.

Definition 3.8 *The* clique-node matrix *of a graph G is a $0,1$ matrix whose columns are indexed by the nodes of G and whose rows are the incidence vectors of the maximal cliques of G.*

Let J be the square matrix all of whose entries are 1 and let I be the identity matrix.

Theorem 3.9 *Let M be a 0,1 clutter matrix with no column of zeroes. The following statements are equivalent:*

(i) *M is a clique-node matrix;*

(ii) *If $\begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \end{pmatrix}$ is a submatrix of M, say with columns $j_1, \dots, j_p$, then M contains a row i such that $m_{ij_k} = 1$ for $k = 1, \dots, p$;*

(iii) *If $J - I$ is a $p \times p$ submatrix of M, where $p \geq 3$, then M contains a row i such that $m_{ij} = 1$ for every column j of $J - I$.*

Proof: (i) $\Rightarrow$ (ii): In the graph G with clique-node matrix M, the nodes $j_1, \ldots, j_p$ are pairwise adjacent and therefore they form a clique. So, a row i as claimed in (ii) must exist.

(ii) $\Rightarrow$ (iii): The first three rows of $J - I$ satisfy the condition in (ii). Therefore, there exists a row i such that $m_{ij} = 1$ for every column j of $J - I$.

(iii) $\Rightarrow$ (i): Assume that (i) does not hold. We show that (iii) does not hold. Let $G(M)$ be the graph having a node for each column of M and an edge between nodes j and k if M has a row i with $m_{ij} = m_{ik} = 1$. Since (i) does not hold, there exists a clique K of $G(M)$ such that, for every row i, $K \not\subseteq N_i \equiv \{j : m_{ij} = 1\}$. Choose K to be minimal with this property. Then, for each $j \in K$, there exists a distinct row i_j such that $K \cap N_{i_j} = K - \{j\}$. This yields a $J - I$ submatrix of M which contradicts (iii). $\Box$

Statement (ii) in the above theorem and the next corollary were noted by Conforti [38].

Corollary 3.10 *There exists a polynomial algorithm to check whether a 0,1 matrix is a clique-node matrix.*

Proof: By Theorem 3.9(ii), one only needs to check every triplet of rows. $\Box$

Theorem 3.11 *Let M be a 0,1 clutter matrix with no column of zeroes. Then M is a perfect matrix if and only if it is the clique-node matrix of a perfect graph.*

Proof: If M is not a clique-node matrix, then Theorem 3.9 implies the existence of a $p \times p$ submatrix $J - I$, $p \geq 3$. It is easy to see that $x_j = 1/(p-1)$ if j is a column of $J - I$, 0 otherwise, is a vertex of $P(M)$. Therefore M is not perfect. Conversely, let M be the clique-node matrix of a pefect graph. Theorem 3.4(ii) implies that M is a perfect matrix. $\Box$

Corollary 3.12 *Let M be a $0,1$ matrix with no column of zeroes. The polytope $P(M) = \{x \in R^n_+ : Mx \leq \mathbf{1}\}$ is integral if and only if the linear system $\{x \geq 0,\ Mx \leq \mathbf{1}\}$ is TDI.*

Exercise 3.13 *Prove Corollary 3.12 using Exercise 3.6.*

This shows (ii) $\Leftrightarrow$ (i) in Theorem 3.1.

3.3 Antiblocker

Let $\mathcal{C}$ be a clutter. The *antiblocker* of $\mathcal{C}$, denoted by $a(\mathcal{C})$ is the clutter such that $V(a(\mathcal{C})) \equiv V(\mathcal{C})$ and $E(a(\mathcal{C}))$ is the set of maximal members of $\{S \subseteq V(\mathcal{C}) : |S \cap T| \leq 1 \text{ for all } T \in E(\mathcal{C})\}$.

Remark 3.14 *For a clutter $\mathcal{C}$, let $G(\mathcal{C})$ be the graph with node set $V(\mathcal{C})$ where two nodes are adjacent if and only if they are both contained in some edge of $\mathcal{C}$. Then $a(\mathcal{C})$ is the family of maximal stable sets of $G(\mathcal{C})$.*

Exercise 3.15 *Prove Remark 3.14.*

In contrast to Proposition 1.15 stating that $b(b(\mathcal{C})) = \mathcal{C}$, in general $a(a(\mathcal{C})) \neq \mathcal{C}$ as shown by the following example.

Exercise 3.16 *Let $M = J - I$ and let $\mathcal{C} \equiv \mathcal{C}(M)$ be the corresponding clutter. Find $a(\mathcal{C})$ and $a(a(\mathcal{C}))$.*

Theorem 3.17 *$a(a(\mathcal{C})) = \mathcal{C}$ if and only if $\mathcal{C}$ is the clutter of maximal cliques of $G(\mathcal{C})$.*

Exercise 3.18 *Prove Theorem 3.17 using Theorem 3.9, Remark 3.14 and Exercise 3.16.*

Exercise 3.19 *Let $\mathcal{C}$ be a clutter. Show that if $M(\mathcal{C})$ is perfect, then $a(a(\mathcal{C})) = \mathcal{C}$.*

3.4 Minimally Imperfect Graphs

Definition 3.20 *A graph is* minimally imperfect *if it is not perfect but every proper node induced subgraph is.*

It follows from the perfect graph theorem (Theorem 3.4) that if G is minimally imperfect, then the complement $\bar{G}$ is also minimally imperfect. A *hole* is a chordless cycle of length at least 4. A hole is *odd* if it contains an odd number of edges. It is easy to check that an odd hole is minimally imperfect. Odd holes and their complements (the *odd antiholes*) are the only known minimally imperfect graphs. Berge [5] proposed the following conjecture, known as the *strong perfect graph conjecture*.

Conjecture 3.21 (Strong Perfect Graph Conjecture) (Berge [5]) *The only minimally imperfect graphs are the odd holes and the odd antiholes.*

Chvátal [32] proposed a weaker conjecture, called the *skew partition conjecture* as a first step towards proving Berge's strong perfect graph conjecture. A graph has a *skew partition* if its nodes can be partitioned into four nonempty sets A, B, C, D such that there are all possible edges between A and B and no edges from C to D. It is easy to verify that odd holes and odd antiholes do not have a skew partition. Therefore Conjecture 3.21 implies the following.

Conjecture 3.22 (Skew Partition Conjecture) (Chvátal [32]) *No minimally imperfect graph has a skew partition.*

In the remainder of this section, we present several known properties of minimally imperfect graphs.

Definition 3.23 *Let α and ω be integers greater than one. A graph G is called an (α, ω)-graph (or partitionable graph) if G has exactly $n = \alpha\omega + 1$ nodes and, for each node $v \in V$, $G \setminus v$ can be partitioned into both α cliques of size ω and ω stable sets of size α.*

Remark 3.24 *If G is an (α, ω)-graph, then α and ω are the stability number and clique number of G.*

Exercise 3.25 *Prove Remark 3.24.*

An example of an (α, ω)-graph is the (α, ω)-web $W_{\alpha\omega}$ constructed as follows: $V(W_{\alpha\omega}) = \{v_0, \cdots, v_{\alpha\omega}\}$ and nodes v_i, v_j are adjacent if and only if $i - j \in \{-\omega + 1, \ldots, -1, 1, \ldots, \omega - 1\} \pmod{\alpha\omega + 1}$.

Theorem 3.26 (Lovász [137]) *If G is minimally imperfect, then G is an (α, ω)-graph.*

We give a proof of this result due to Gasparyan [95]. The proof uses a result of Bridges and Ryser [19]:

Theorem 3.27 *Let Y and Z be $n \times n$ 0,1 matrices such that $YZ = J - I$. Then*

(i) each row and column of Y has the same number r of ones, each row and column of Z has the same number s of ones, with $rs = n - 1$,

(ii) $YZ = ZY$;

(iii) For each $j = 1, \ldots, n$, there exist s rows of Y that sum up to $\mathbf{1} - e^j$ where e^j denotes the j^{th} unit row vector.

Proof: It is straightforward to check that $(J - I)^{-1} = \frac{1}{n-1}J - I$. Hence

$$YZ = J - I \Rightarrow YZ(\frac{1}{n-1}J - I) = I \Rightarrow Z(\frac{1}{n-1}J - I)Y = I$$

$$\text{i.e.} \qquad ZY = \frac{1}{n-1}ZJY - I = \frac{1}{n-1}\mathbf{s}\mathbf{r}^T - I$$

where $\mathbf{s} \equiv Z\mathbf{1}$ and $\mathbf{r} \equiv Y^T\mathbf{1}$.

It follows that, for each i and j, $n - 1$ divides $r_i s_j$. On the other hand, the trace of the matrix ZY is equal to the trace of YZ, which is 0. As ZY is a nonnegative matrix, it follows that it has 0's in its main diagonal. Hence $r_i s_i = n - 1$ for all i. Now consider distinct i, j. Since $r_i s_i = r_j s_j = n - 1$ and $n - 1$ divides $r_i s_j$, it follows that $r_i = r_j$ and $s_i = s_j$. Therefore, all columns of Z have the same sum s and all rows of Y have the same sum r. Furthermore, $ZY = J - I$ and, by symmetry, all columns of Y have the same sum and all rows of Z have the same sum. (iii) follows from $Y^T Z^T = J - I$. □

Proof of Theorem 3.26: Let G be a minimally imperfect graph with n nodes. Let $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Then G satisfies

$$\omega = \chi(G \backslash v) \text{ for every node } v \in V$$

$$\text{and } \ \omega = \omega(G \backslash S) \text{ for every stable set } S \subseteq V.$$

Let A_0 be an α-stable set of G. Fix an ω-coloring of each of the α graphs $G \backslash s$ for $s \in A_0$, let $A_1, \ldots, A_{\alpha\omega}$ be the stable sets occuring as a color-class in one of these colorings and let $\mathcal{A} := \{A_0, A_1, \ldots, A_{\alpha\omega}\}$. Let $\mathbf{A}$ be the corresponding stable set versus node incidence matrix. Define $\mathcal{B} := \{B_0, B_1, \ldots, B_{\alpha\omega}\}$ where B_i is an ω-clique of $G \backslash A_i$. Let $\mathbf{B}$ be the corresponding clique versus node incidence matrix.

Claim: Every ω-clique of G intersects all but one of the stables sets in $\mathcal{A}$.

Proof: Let $S_1, \ldots, S_\omega$ be any ω-coloring of $G \setminus v$. Since any ω-clique C of G has at most one node in each S_i, C intersects all S_i's if $v \notin C$ and all but one if $v \in C$. Since C has at most one node in A_0, the claim follows.

In particular, it follows that $\mathbf{AB}^T = J - I$. By Theorem 3.27(i) $n = \alpha\omega + 1$ and by Theorem 3.27(iii) the nodes of $G \backslash j$ can be partitioned into ω stable sets of size α and the nodes of $G \setminus j$ can be partitioned into α cliques of size ω. So G is an (α, ω)-graph. $\Box$

It follows from the definition of a partitionable graph that its complement is also partitionable. Therefore Theorem 3.26 implies the perfect graph theorem.

Corollary 3.28 (Perfect Graph Theorem) (Lovász [136])
G is perfect if and only if $\bar{G}$ is perfect.

Exercise 3.29 *Prove the Perfect Graph Theorem using Theorem 3.26.*

Let $\mathcal{C}_k$ and $\mathcal{S}_k$ denote respectively the clutters of k-cliques and k-stable sets of G. $\mathbf{S}_k$ and $\mathbf{C}_k$ are respectively the incidence matrices of k-stable sets versus nodes and k-cliques versus nodes of G. Using Theorem 3.26, Padberg [154] proved that a minimally imperfect graph G has exactly n maximum cliques and n maximum stable sets, and that the rows of $\mathbf{S}_\alpha$ and $\mathbf{C}_\omega$ can be permuted so that $\mathbf{S}_\alpha \mathbf{C}_\omega^T = J - I$. Bland et al. [17] proved that the statement remains true for (α, ω)-graphs and Chvátal et al. [34] observed that the converse is also true. These results are included in the next theorem.

Theorem 3.30 *Let G be a graph with n nodes and let $\alpha > 1$, $\omega > 1$ be integers. The following are equivalent:*

1) G is an (α, ω)-graph.

2) $\alpha = \alpha(G)$ and, for every node $v \in V$ and stable set $S \subseteq V$, $\omega = \omega(G \backslash S) = \chi(G \backslash v)$.

3) $\mathbf{S}_\alpha$ and $\mathbf{C}_\omega$ are $n \times n$ matrices and their rows can be permuted so that $\mathbf{S}_\alpha \mathbf{C}_\omega^T = J - I$.

Proof: 1)⇒2) : Let G be an (α, ω)-graph. It follows from the definition that, for every node $v \in V$, $\chi(G \backslash v) = \omega$. By Remark 3.24, $\omega(G) = \omega$, so 2) holds when $S = \emptyset$. Now assume $S \neq \emptyset$ and let $x \in S$. Consider a partition of $G \backslash x$ into α cliques of size ω. As $|S| \leq \alpha$, it follows that one of the cliques of size ω of this partition is disjoint from S. Hence, $\omega(G \backslash S) = \omega$.

2)⇒3) : This follows from the proof of Theorem 3.26. Indeed, this proof shows that if G satisfies 2), then $\mathbf{A}\mathbf{B}^T = J - I$. Now, as $\mathbf{A}$ is nonsingular, it follows that for each i, $\mathbf{A}x = \mathbf{1} - e^i$ has a unique solution. Hence $\mathcal{B} = \mathcal{C}_\omega$ by the Claim in the proof of Theorem 3.26. As A_0 is an arbitrary α-stable set and $\mathbf{B}$ is nonsingular, it follows that $\mathcal{A} = \mathcal{S}_\alpha$. Thus $\mathbf{S}_\alpha \mathbf{C}_\omega^T = J - I$.

3)⇒1) : By Theorem 3.27, $n = \alpha\omega + 1$, there are α cliques of size ω that partition the nodes of $G \backslash v_j$ and there are ω stable sets of size α that partition the nodes of $G \backslash v_j$. □

A 0,1 matrix M with no column of zeroes is *minimally imperfect* if it is not perfect but all its column submatrices are.

Theorem 3.31 (Padberg [154]) *Let M be a minimally imperfect 0,1 matrix. Then it has a non-singular row submatrix $\bar{M}$ with exactly r ones in every row and column. Moreover, rows of M not in $\bar{M}$ have at most $r - 1$ ones.*

Proof: If M is not a clique-node matrix, the result follows from Theorems 3.9(iii).

If M is the clique-node matrix of a graph G, then G is minimally imperfect by Theorem 3.11. Now G has exactly $|V(G)|$ maximum cliques by Theorems 3.26 and 3.30 3). $\square$

Corollary 3.32 *Let M be a $0,1$ matrix with no column of zeroes. The polytope $P(M) = \{x \in R_+^n : Mx \leq \mathbf{1}\}$ is integral if and only if $max\{wx : x \in P(M)\}$ has an integral optimal solution for all $w \in \{0,1\}^n$.*

This is surprising since, in general, we need integral optimal solutions for all $w \in Z^n$ to conclude that a polytope is integral.

Exercise 3.33 *Prove Corollary 3.32.*

This proves (iii) $\Leftrightarrow$ (ii) in Theorem 3.1.

Next, we prove properties of partitionable graphs following [45]. A graph G has a *star cutset* if there exists a node set S consisting of a node and some of its neighbors such that $G \setminus S$ is disconnected.

Theorem 3.34 *An (α, ω)-graph G with n nodes has the following properties:*

1) *[154][17] G has exactly n ω-cliques and n α-stable sets, which can be indexed as $C_1, \ldots, C_n$ and $S_1, \ldots, S_n$, so that $C_i \cap S_j$ is empty if and only if $i = j$. We say that S_i and C_i are mates.*

2) *[154][17] Every $v \in V$ belongs to exactly α α-stable sets and their intersection contains no other node.*

3) *[154][17] For every $v \in V$, $G \setminus v$ has a unique ω-coloring and its color classes are the α-stable sets that are mates of the ω-cliques containing v.*

4) *[34] If $e \in E(G)$ does not belong to any ω-clique, then $G \setminus e$ is an (α, ω)-graph.*

5) *[199] Let G_ω be the intersection graph of all the ω-cliques of G. Then G_ω is an (α, ω)-graph.*

6) *[17] Let S_1, S_2 be two α-stable sets. Then the graph induced by $S_1 \Delta S_2$ is connected.*

7) *[32] G contains no star cutset.*

8) *[177] Any proper induced subgraph H of G with $\omega(H) = \omega$ has at most $|V(H)| - \omega + 1$ ω-cliques.*

9) *[177] Let (V_1, V_2) be a partition of $V(G)$ such that both V_1 and V_2 contain at least one ω-clique. Then the number of ω-cliques that intersect both V_1 and V_2 is at least $2\omega - 2$.*

10) *[177] G is $(2\omega - 2)$-node connected.*

11) *[143] $e \in E(G)$ belongs to $\omega - 1$ ω-cliques if and only if $\alpha(G \setminus e) > \alpha$.*

Proof: 1) follows from Theorem 3.30 3).

2) follows Theorem 3.27 and Theorem 3.30 3).

3) From Theorem 3.27 and Theorem 3.30 3), we have that $\mathbf{S}_\alpha^T \mathbf{C}_\omega$ is a 0,1 matrix. Hence if two ω-cliques intersect, their mates are disjoint and viceversa. It follows that the mates of ω-cliques containing v partition $G \setminus v$. To show that this partition is unique, just notice that there is a one-to-one correspondence between ω-colorings of $G \setminus v$ and 0,1 solutions of $\mathbf{S}_\alpha^T x = \mathbf{1} - e^v$. As $\mathbf{S}_\alpha$ is non-singular, it follows that $G \setminus v$ has a unique ω-coloring.

4) follows from the definition of (α, ω)-graphs.

5) It follows from 2) that $\mathbf{C}_\omega^T$ and $\mathbf{S}_\alpha^T$ are respectively the ω-cliques versus nodes and α-stable sets versus nodes incidence matrices of G_ω. So 5) follows from Theorem 3.30 3).

6) Suppose the graph $G(S_1 \Delta S_2)$ is disconnected and let $S \subset S_1 \Delta S_2$ induce one of its connected components. For $i = 1, 2$, let $S_i' = S \cap S_i$ and $S_i'' = S_i - S_i'$. Then $S' = S_1' \cup S_2''$ and $S'' = S_2' \cup S_1''$ are stable sets of G. As $|S'| + |S''| = 2\alpha$, it follows that S' and S'' are α-stable sets. Let C_1 be the mate of S_1. Then C_1 meets both S' and S''. As C_1 is

disjoint from S_1, it follows that C_1 meets both S_2' and S_2''. But this is a contradiction, as $S_2' \cup S_2'' = S_2$ is a stable set.

7) Let U, V_1, V_2 be a partition of V such that U is a star cutset of G and V_1 induces a connected component of $G \setminus U$. Let G_1 and G_2 be the graphs induced by $U \cup V_1$ and $U \cup V_2$ respectively, and let $u \in U$ be adjacent to all the nodes in U. Finally, let S_i be the color class of an ω-coloring of G_i containing u, where $i \in \{1, 2\}$. Then S_i meets all the ω-cliques of G_i, i.e. $\omega(G \setminus \{S_1 \cup S_2\}) < \omega$. On the other hand, $S_1 \cup S_2$ is a stable set, a contradiction to Theorem 3.30 2).

8) Let $\mathcal{S} = \{S_1, \ldots, S_\omega\}$ be the color classes of an ω-coloring of H. Then $\mathbf{CS}^T = J$, where $\mathbf{C}$ denotes the incidence matrix of maximum cliques versus nodes of H. Since $\mathbf{C}_\omega$ has full column rank, $\mathbf{C}$ also has full column rank. As $rk(\mathbf{S}) = \omega$ and $rk(J) = 1$, it follows from linear algebra that $|\mathcal{C}| = rk(\mathbf{C}) \leq |V(H)| - \omega + 1$.

We leave the proofs of 9) and 10) as an exercise.

11) If $e = uv$ belongs to $\omega - 1$ ω-cliques $C_1, \ldots, C_{\omega-1}$ then, by 2), there exist an ω-clique C_0 containing u but not v and an ω-clique C_ω containing v but not u. Let $S_0, \ldots, S_\omega$ be their mates. By 3), each of $S_0, \ldots, S_{\omega-1}$ and $S_1, \ldots, S_\omega$ covers $\alpha\omega$ nodes. Since $n = \alpha\omega + 1$, S_0 and S_ω have $\alpha - 1$ nodes in common and therefore their union is a stable set of $G \setminus e$. Conversely, let S be an $(\alpha + 1)$-stable set of $G \setminus e$ and let $e = uv$. $S - \{u\}$ and $S - \{v\}$ are α-stable sets of G. So, by 1), all the ω-cliques containing u also contain v, except for the mate of $S - \{u\}$. By 3), there are $\omega - 1$ such ω-cliques. □

Exercise 3.35 *Prove 9) and 10) in Theorem 3.34. Hint: Show that 8) implies 9). Then show that 2) plus 9) imply 10).*

Note that Theorem 3.34 7) is a special case of the Skew Partition Conjecture: indeed, a star cutset is a skew partition A, B, C, D where A or B has cardinality one.

Chapter 4

Ideal Matrices

A $0, 1$ clutter matrix A is *ideal* if the polyhedron $Q(A) \equiv \{x \geq 0 : Ax \geq \mathbf{1}\}$ is integral. Ideal matrices give rise to set covering problems that can be solved as linear programs, for all objective functions.

This concept was introduced by Lehman under the name of width-length property. Lehman [132] showed that ideal 0,1 matrices always come in pairs (Theorem 1.17: A is ideal if and only if its blocker $b(A)$ is ideal) and that the width-length inequality is in fact a characterization of idealness (Theorem 1.21). Another important result of Lehman about ideal 0,1 matrices is the following.

Theorem 4.1 (Lehman [133]) *For a 0,1 matrix A, the following statements are equivalent:*

(i) the matrix A is ideal,

(ii) min $\{wx : \ Ax \geq \mathbf{1}, \ x \geq 0\}$ has an integral optimal solution x for all $w \in \{0, 1, +\infty\}^n$.

The fact that (i) implies (ii) is an immediate consequence of the definition of idealness. The difficult part of Lehman's theorem is that (ii) implies (i). The main purpose of this chapter is to prove this result. This is done by studying properties of minimally nonideal matrices.

4.1 Minimally Nonideal Matrices

A 0,1 clutter matrix A is *minimally nonideal* (*mni*) if

(i) $Q(A) \equiv \{x \geq 0 : \ Ax \geq \mathbf{1}\}$ is not an integral polyhedron,

(ii) For every $i = 1, \ldots, n$, both $Q(A) \cap \{x : \ x_i = 0\}$ and $Q(A) \cap \{x : x_i = 1\}$ are integral polyhedra.

If A is *mni*, the clutter $\mathcal{C}(A)$ is also called *mni*. Equivalently, a clutter $\mathcal{C}$ is *mni* if it is not ideal but all its proper minors are ideal.

For $t \geq 2$ integer, let $\mathcal{J}_t$ denote the clutter with $t+1$ vertices and edges corresponding, respectively, to the points and lines of the finite degenerate projective plane. Namely, $V(\mathcal{J}_t) \equiv \{0, \ldots, t\}$, and $E(\mathcal{J}_t) \equiv \{\{1, \ldots, t\}, \{0, 1\}, \{0, 2\}, \ldots, \{0, t\}\}$.

Exercise 4.2 *Show that $\mathcal{J}_t$ is minimally nonideal.*

A matrix A is *isomorphic* to a matrix B if B can be obtained from A by a permutation of rows and a permutation of columns.

Let J denote a square matrix all of whose entries are 1's, and let I be the identity matrix. Given a *mni* matrix A, let $\bar{x}$ be an extreme point of the polyhedron $Q(A) \equiv \{x \geq 0 : \ Ax \geq \mathbf{1}\}$ with fractional components. The maximum row submatrix $\bar{A}$ of A such that $\bar{A}\bar{x} = \mathbf{1}$ is called a *core* of A. So A has one core for each fractional extreme point of $Q(A)$.

Theorem 4.3 (Lehman [133]) *Let A be a mni matrix and $B = b(A)$. Then*

(i) A has a unique core $\bar{A}$ and B has a unique core $\bar{B}$;

(ii) $\bar{A}$ and $\bar{B}$ are square matrices;

(iii) Either A is isomorphic to $M(\mathcal{J}_t)$, $t \geq 2$, or the rows of $\bar{A}$ and $\bar{B}$ can be permuted so that

$$\bar{A}\bar{B}^T = J + dI$$

for some positive integer d.

Lehman's proof of this theorem is rather terse. Seymour [189], Padberg [157] and Gasparyan, Preissmann and Sebö [96] give more accessible presentations of Lehman's proof. In the next section, we present a proof of Lehman's theorem following Padberg's polyhedral point of view.

Bridges and Ryser [19] studied square matrices Y, Z that satisfy the matrix equation $YZ = J + dI$.

Theorem 4.4 (Bridges and Ryser [19]) *Let Y and Z be $n \times n$ 0,1 matrices such that $YZ = J + dI$ for some positive integer d. Then*

(i) each row and column of Y has the same number r of ones, each row and column of Z has the same number s of ones with $rs = n + d$,

(ii) $YZ = ZY$,

(iii) For each $j = 1, \dots, n$, there exist s rows of Y that sum up to $\mathbf{1} + de^j$, where e^j denotes the j^{th} unit row vector. This set of s rows is given by the j^{th} row of Z viewed as a characteristic vector.

Proof: It is straightforward to check that $(J+dI)^{-1} = \frac{1}{d}I - \frac{1}{d(n+d)}J$. Hence

$$YZ = J+dI \Rightarrow YZ(\frac{1}{d}I - \frac{1}{d(n+d)}J) = I \Rightarrow Z(\frac{1}{d}I - \frac{1}{d(n+d)}J)Y = I$$

$$\text{i.}e. \qquad ZY = \frac{1}{n+d}ZJY + dI = \frac{1}{n+d}\mathbf{s}\mathbf{r}^T + dI$$

where $\mathbf{s} \equiv Z\mathbf{1}$ and $\mathbf{r} \equiv Y^T\mathbf{1}$.

It follows that, for each i and j, $n + d$ divides $r_i s_j$. On the other hand, the trace of the matrix ZY is equal to the trace of YZ, which is $n(d+1)$. This implies $\frac{1}{n+d}(\sum_1^n s_i r_i) = n$ and, since $s_i > 0$ and $r_i > 0$, we have $r_i s_i = n + d$. Now consider distinct i, j. Since $r_i s_i = r_j s_j = n + d$ and $n + d$ divides $r_i s_j$ and $r_j s_i$, it follows that $r_i = r_j$ and $s_i = s_j$. Therefore, all columns of Z have the same sum s and all rows of Y have the same sum r. Furthermore, $ZY = J + dI$ and, by symmetry, all columns of Y have the same sum and all rows of Z have the same sum. (iii) follows from $Y^T Z^T = J + dI$. $\Box$

Theorems 4.3 and 4.4 have the following consequence.

Corollary 4.5 *Let A be a mni matrix nonisomorphic to $M(\mathcal{J}_t)$. Then it has a non-singular row submatrix $\bar{A}$ with exactly r ones in every row and column. Moreover, rows of A not in $\bar{A}$ have at least $r+1$ ones.*

This implies the next result, which is a restatement of Theorem 4.1.

Corollary 4.6 *Let A be a $0,1$ matrix. The polyhedron $Q(A) = \{x \in R^n_+ : Ax \geq \mathbf{1}\}$ is integral if and only if $min\{wx : x \in Q(A)\}$ has an integral optimal solution for all $w \in \{0, 1, \infty\}^n$.*

Note the similarity with Corollary 3.32.

Exercise 4.7 *Prove Corollary 4.6 from Theorem 4.3 and Corollary 4.5.*

Exercise 4.8 *Prove Theorem 1.8 from Corollary 4.6.*

4.1.1 Proof of Lehman's Theorem

Let A be an $m \times n$ *mni* matrix, $\bar{x}$ a fractional extreme point of $Q(A) \equiv \{x \in R^n_+ :\ Ax \geq \mathbf{1}\}$ and $\bar{A}$ a core of A. That is, $\bar{A}$ is the maximal row submatrix of A such that $\bar{A}\bar{x} = \mathbf{1}$. For simplicity of notation, assume that $\bar{A}$ corresponds to the first p rows of A, i.e. the entries of $\bar{A}$ are a_{ij} for $i = 1, \ldots, p$ and $j = 1, \ldots, n$. Since A is *mni*, every component of $\bar{x}$ is nonzero. Therefore $p \geq n$ and $\bar{A}$ has no row or column containing only 0's or only 1's.

The following easy result will be applied to the bipartite representation G of the 0,1 matrix $J - \bar{A}$ where J denotes the $p \times n$ matrix of all 1's, namely ij is an edge of G if and only if $a_{ij} = 0$, for $1 \leq i \leq p$ and $1 \leq j \leq n$. Let $d(u)$ denote the degree of node u.

Lemma 4.9 (de Bruijn and Erdös [71]) *Let $(I \cup J, E)$ be a bipartite graph with no isolated node. If $|I| \geq |J|$ and $d(i) \geq d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$, then $|I| = |J|$ and $d(i) = d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$.*

Proof:
$|I| = \sum_{i \in I}(\sum_{j \in N(i)} \frac{1}{d(i)}) \leq \sum_{i \in I} \sum_{j \in N(i)} \frac{1}{d(j)} = \sum_{j \in J} \sum_{i \in N(j)} \frac{1}{d(j)} = |J|$. Now the hypothesis $|I| \geq |J|$ implies that equality holds throughout. So $|I| = |J|$ and $d(i) = d(j)$ for all $i \in I$, $j \in J$ such that $ij \in E$. $\square$

The key to proving Lehman's theorem is the following lemma.

Lemma 4.10 *$p = n$ and, if $a_{ij} = 0$ for $1 \leq i, j \leq n$, then row i and column j of $\bar{A}$ have the same number of ones.*

Proof: Let x^j be defined by

$$x_k^j = \begin{cases} \bar{x}_k & \text{if} \quad k \neq j \\ 1 & \text{if} \quad k = j \end{cases}$$

and let F_j be the face of $Q(A) \cap \{x_j = 1\}$ of smallest dimension that contains x^j. Since A is *mni*, F_j is an integral polyhedron. The proof of the lemma will follow unexpectedly from computing the dimension of F_j.

The point x^j lies at the intersection of the hyperplanes in $\bar{A}x = \mathbf{1}$ such that $a_{kj} = 0$ (at least $n - \sum_{k=1}^{p} a_{kj}$ such hyperplanes are independent since $\bar{A}$ has rank n) and of the hyperplane $x_j = 1$ (independent of the previous hyperplanes). It follows that

$$dim(F_j) \leq n - (n - \sum_{k=1}^{p} a_{kj} + 1) = \sum_{k=1}^{p} a_{kj} - 1$$

Choose a row a^i of $\bar{A}$ such that $a_{ij} = 0$. Since $x^j \in F_j$, it is greater than or equal to a convex combination of extreme points b^ℓ of F_j, say $x^j \geq \sum_{\ell=1}^{t} \gamma_\ell b^\ell$, where $\gamma > 0$ and $\sum \gamma_\ell = 1$.

$$1 = a^i x^j \geq \sum_{\ell=1}^{t} \gamma_\ell a^i b^\ell \geq 1 \tag{4.1}$$

Therefore, equality must hold throughout. In particular $a^i b^\ell = 1$ for $\ell = 1, \ldots, t$. Since b^ℓ is a 0,1 vector, it has exactly one nonzero entry in the set of columns k where $a_{ik} = 1$. Another consequence of the fact that equality holds in (4.1) is that $x_k^j = \sum_{\ell=1}^{t} \gamma_\ell b_k^\ell$ for every k where $a_{ik} = 1$. Now, since $x_k^j > 0$ for all k, it follows that F_j contains at least $\sum_{k=1}^{n} a_{ik}$ linearly independent points b^ℓ, i.e.

$$dim(F_j) \geq \sum_{k=1}^{n} a_{ik} - 1.$$

Therefore, $\sum_{k=1}^{n} a_{ik} \leq \sum_{k=1}^{p} a_{kj}$ for all i, j such that $a_{ij} = 0$.

Now Lemma 4.9 applied to the bipartite representation of $J - \bar{A}$ implies that $p = n$ and

$$\sum_{k=1}^{n} a_{ik} = \sum_{k=1}^{n} a_{kj} \text{ for all } i, j \text{ such that } a_{ij} = 0.$$

□

Lemma 4.11 *$\bar{x}$ has exactly n adjacent extreme points in $Q(A)$, all with 0,1 coordinates.*

Proof: By Lemma 4.10, exactly n inequalities of $A\bar{x} \geq \mathbf{1}$ are tight, namely $\bar{A}\bar{x} = \mathbf{1}$. In the polyhedron $Q(A)$, an edge adjacent to $\bar{x}$ is defined by $n-1$ of the n equalities in $\bar{A}x = \mathbf{1}$. Moving along such an edge from $\bar{x}$, at least one of the coordinates decreases. Since $Q(A) \in R^n_+$, this implies that $\bar{x}$ has exactly n adjacent extreme points on $Q(A)$. Suppose $\bar{x}$ has a fractional adjacent extreme point $\bar{x}'$. Since A is *mni*, $0 < \bar{x}'_j < 1$ for all j. Let $\bar{A}'$ be the $n \times n$ nonsingular submatrix of A such that $\bar{A}'\bar{x}' = \mathbf{1}$. Since $\bar{x}$ and $\bar{x}'$ are adjacent on $Q(A)$, $\bar{A}$ and $\bar{A}'$ differ in only one row. W.l.o.g. assume that $\bar{A}'$ corresponds to rows 2 to $n+1$. Since A contains no dominating row, there exists j such that $a_{1j} = 0$ and $a_{n+1,j} = 1$. Since $\bar{A}'$ cannot contain a column with only 1's, $a_{ij} = 0$ for some $2 \leq i \leq n$. But now, Lemma 4.9 is contradicted with row i and column j in either $\bar{A}$ or $\bar{A}'$. □

Lemma 4.11 has the following implication. Let $\bar{B}$ denote the $n \times n$ 0,1 matrix whose rows are the extreme points of $Q(A)$ adjacent to $\bar{x}$. By Remark 1.16(i), $\bar{B}$ is a submatrix of B. By Lemma 4.11, $\bar{B}$ satisfies the matrix equation

$$\bar{A}\bar{B}^T = J + D$$

where J is the matrix of all 1's and D is a diagonal matrix with positive diagonal entries $d_1, \ldots, d_n$.

Lemma 4.12 *Either*

(i) $\bar{A} = \bar{B}$ are isomorphic to $M(\mathcal{J}_t)$, for $t \geq 2$, or

(ii) $D = dI$, where d is a positive integer.

Proof: Consider the bipartite representation G of the 0,1 matrix $J - \bar{A}$.

Case 1: G is connected.
Then it follows from Lemma 4.10 that

$$\sum_k a_{ik} = \sum_k a_{kj} \text{ for all } i, j. \tag{4.2}$$

Let α denote this common row and column sum.

$$(n + d_1, \ldots, n + d_n) = \mathbf{1}^T(J + D) = \mathbf{1}^T \bar{A}\bar{B}^T = (\mathbf{1}^T \bar{A})\bar{B}^T = \alpha \mathbf{1}^T \bar{B}^T$$

Since there is at most one d, $1 \le d < \alpha$, such that $n + d$ is a multiple of α, all d_i must be equal to d, i.e. $D = dI$.

Case 2: G is disconnected.
Let $q \ge 2$ denote the number of connected components in G and let $\bar{A} = \begin{pmatrix} K_1 & & \mathbf{1} \\ & \ldots & \\ \mathbf{1} & & K_q \end{pmatrix}$ where K_t are 0,1 matrices, for $t = 1, \ldots, q$. It follows from Lemma 4.10 that the matrices K_t are square and $\sum_k a_{ik} = \sum_k a_{kj} = \alpha_t$ in each K_t.

Suppose first that $\bar{A}$ has no row with $n - 1$ ones. Then every K_t has at least two rows and columns. We claim that, for every j, k, there exist i, l such that $a_{ij} = a_{ik} = a_{lj} = a_{lk} = 1$. The claim is true if $q \ge 3$ or if $q = 2$ and j, k are in the same component (simply take two rows i, l from a different component). So suppose $q = 2$, column j is in K_1 and column k is in K_2. Since no two rows are identical, we must have $\alpha_1 \ge 1$, i.e. $a_{ij} = 1$ for some row i of K_1. Similarly, $a_{lk} = 1$ for some row l of K_2. The claim follows.

For each row b of $\bar{B}$, the vector $\bar{A}b^T$ has an entry greater than or equal to 2, so there exist two columns j, k such that $b_j = b_k = 1$. By the claim, there exist rows a_i and a_l of $\bar{A}$ such that $a_i b^T \ge 2$ and $a_l b^T \ge 2$, contradicting the fact that $\bar{A}b^T$ has exactly one entry greater than 1.

Therefore $\bar{A}$ has a row with $n - 1$ ones. Now it is routine to check that $\bar{A}$ is isomorphic to $M(\mathcal{J}_t)$, for $t \ge 2$. $\square$

Exercise 4.13 *Let $\bar{A}$ and $\bar{B}$ be $n \times n$ 0,1 matrices and assume $\bar{A}$ has a row with $n-1$ ones. Show that, up to permutation of rows and columns, $\bar{A} = \bar{B} = M(\mathcal{J}_t)$ is the only solution to $\bar{A}\bar{B}^T = J + D$ when $D > 0$ is a diagonal matrix.*

To complete the proof of Theorem 4.3, it only remains to show that the core $\bar{A}$ is unique and that $\bar{B}$ is a core of B and is unique.

If $\bar{A} = M(\mathcal{J}_t)$ for some $t \geq 2$, then the fact that A has no dominated rows implies that $A = \bar{A}$. Thus $B = \bar{B} = M(\mathcal{J}_t)$. So, the theorem holds in this case.

If $\bar{A}\bar{B}^T = J + dI$ for some positive integer d, then, by Theorem 4.4, all rows of $\bar{A}$ contain r ones. Therefore, $\bar{x}_j = \frac{1}{r}$, for $j = 1, \ldots, n$. The feasibility of $\bar{x}$ implies that all rows of A have at least r ones, and Lemma 4.10 implies that exactly n rows of A have r ones. Now $Q(A)$ cannot have a fractional extreme point $\bar{x}'$ distinct from $\bar{x}$, since the above argument applies to $\bar{x}'$ as well. Therefore A has a unique core $\bar{A}$. Since $\bar{x}$ has exactly n neighbors in $Q(A)$ and they all have s components equal to one, the inequality $\sum_1^n x_i \geq s$ is valid for the 0,1 points in $Q(A)$. This shows that every row of B has at least s ones and exactly n rows of B have s ones. Since B is *mni*, $\bar{B}$ is the unique core of B. $\Box$

4.1.2 Examples of *mni* Clutters

Let $Z_n = \{0, \ldots, n-1\}$. We define addition of elements in Z_n to be addition modulo n. Let $k \leq n-1$ be a positive integer. For each $i \in Z_n$, let C_i denote the subset $\{i, i+1, \ldots, i+k-1\}$ of Z_n. Define the *circulant* clutter $\mathcal{C}_n^k$ by $V(\mathcal{C}_n^k) \equiv Z_n$ and $E(\mathcal{C}_n^k) \equiv \{C_0, \ldots, C_{n-1}\}$.

Lehman [132] gave three infinite classes of minimally nonideal clutters: $\mathcal{C}_n^2$, $n \geq 3$ odd, their blockers, and the degenerate projective planes $\mathcal{J}_n$, $n \geq 2$.

Conjecture 4.14 (Cornuéjols and Novick [65]) *There exists n_0 such that, for $n \geq n_0$, all mni matrices have a core isomorphic to $\mathcal{C}_n^2$, $\mathcal{C}_n^{\frac{n+1}{2}}$ for $n \geq 3$ odd, or $\mathcal{J}_n$, for $n \geq 2$.*

However, there exist several known “small” *mni* matrices that do not belong to any of the above classes. For example, Lehman [132] noted that $\mathcal{F}_7$ is *mni*. $\mathcal{F}_7$ is the clutter with 7 vertices and 7 edges corresponding to points and lines of the Fano plane (finite projective geometry on 7 points):

$$M(\mathcal{F}_7) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 4.15 *Show that $\mathcal{F}_7$ is mni. Show that $b(\mathcal{F}_7) = \mathcal{F}_7$.*

Let K_5 denote the complete graph on five nodes and let $\mathcal{O}_{K_5}$ denote the clutter whose vertices are the edges of K_5 and whose edges are the odd cycles of K_5 (the triangles and the pentagons). Seymour [183] noted that $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, and $\mathcal{C}_9^2$ with the extra edge $\{3, 6, 9\}$ are *mni*.

Exercise 4.16 *Show that $\mathcal{O}_{K_5}$ is mni.*

Ding [75] found the following *mni* clutter: $V(\mathcal{D}_8) \equiv \{1, \ldots, 8\}$, $E(\mathcal{D}_8) \equiv \{\{1, 2, 6\}, \{2, 3, 5\}, \{3, 4, 8\}, \{4, 5, 7\}, \{2, 5, 6\}, \{1, 6, 7\}, \{4, 7, 8\}, \{1, 3, 8\}\}$.

Cornuéjols and Novick [65] characterized the *mni* circulant clutters $\mathcal{C}_n^k$. They showed that the following ten clutters are the only *mni* $\mathcal{C}_n^k$ for $k \geq 3$:

$$\mathcal{C}_5^3,\ \mathcal{C}_8^3,\ \mathcal{C}_{11}^3,\ \mathcal{C}_{14}^3,\ \mathcal{C}_{17}^3,\ \mathcal{C}_7^4,\ \mathcal{C}_{11}^4,\ \mathcal{C}_9^5,\ \mathcal{C}_{11}^6,\ \mathcal{C}_{13}^7.$$

Independently, Qi [163] discovered $\mathcal{C}_9^5$ and $\mathcal{C}_{11}^6$ and Ding [75] discovered $\mathcal{C}_8^3$.

Let $\mathcal{T}_{K_5}$ denote the clutter whose vertices are the edges of K_5 and whose edges are the triangles of K_5 (interestingly, $M(\mathcal{T}_{K_5})$ is also the node-node adjacency matrix of the Petersen graph). It can be shown that $\mathcal{T}_{K_5}$, $core(b(\mathcal{T}_{K_5}))$ and their blockers are *mni*. Often, when a *mni* clutter $\mathcal{H}$ has the property that $core(\mathcal{H})$ and $core(b(\mathcal{H}))$ are also *mni*, many more *mni* clutters can be constructed from $\mathcal{H}$ and from $b(\mathcal{H})$, see [65]. For example, Cornuéjols and Novick [65] have constructed more than one thousand mni clutters from $\mathcal{T}_{K_5}$. More results can be found in [149].

Exercise 4.17 *Let $\mathcal{T}'$ be the clutter obtained from $\mathcal{T}_{K_5}$ as follows. $V(\mathcal{T}') = V(\mathcal{T}_{K_5})$ and $E(\mathcal{T}') = E(\mathcal{T}_{K_5}) \cup \{P\}$ where P denotes a set of 5 edges of K_5 that form a pentagon. Show that $\mathcal{T}'$ is minimally nonideal.*

Lütolf and Margot [142] designed a computer program that enumerates possible cores of minimally nonideal matrices. It first enumerates the square 0,1 matrices Y, Z that satisfy the matrix equation $YZ = J + dI$, and then checks that the covering polyhedron has a unique fractional extreme point. Lütolf and Margot [142] enumerated all square *mni* matrices of dimension at most 12×12 and found 20 such matrices (previously, only 15 were known); they found 13 new square *mni* matrices of dimensions 14×14 and 17×17; and they found 38 new nonsquare *mni* matrices with 11, 14 and 17 columns with nonisomorphic cores. The overwhelming majority of these examples have $d = 1$: Only three cores with $d = 2$ are known (namely $\mathcal{F}_7, \mathcal{T}_{K_5}$ and the core of its blocker) and none with $d \geq 3$.

Theorem 4.18 (Cornuéjols, Guenin, Margot [64]) *Let A be a mni matrix nonisomorphic to $M(\mathcal{J}_t)$, $t \geq 2$. If A is minimally nonpacking, then $d = 1$.*

Conjecture 4.19 ([64]) *Let A be a mni matrix nonisomorphic to $M(\mathcal{J}_t)$, $t \geq 2$. Then A is minimally nonpacking if and only if $d = 1$.*

Using a computer program, this conjecture was verified for all known minimally nonideal matrices with $n \leq 14$.

Proof of Theorem 4.18: We show that, if $\mathcal{C} \neq \mathcal{J}_t$ is a mni clutter with $d > 1$ (i.e. $rs > n + 1$ using the notation of Theorem 4.4), then $\mathcal{C}$ is not minimally nonpacking.

Let L be an edge of $\bar{\mathcal{C}} \equiv core(\mathcal{C})$ and let U be the unique edge of $core(b(\mathcal{C}))$ such that $|L \cap U| > 1$ (U is called the *mate* of L). Let i be any vertex in $L \cap U$ and let $I = (L - U) \cup \{i\}$.

Claim 1: Every transversal of $\mathcal{C} \setminus I$ has cardinality at least $s - 1$.

Proof: It suffices to show that every transversal of $\bar{\mathcal{C}} \setminus I$ has cardinality at least $s - 1$. Suppose there exists a transversal T of $\bar{\mathcal{C}} \setminus I$ with $|T| \leq s - 2$. Let j be any vertex in $U - \{i\}$. By Theorem 4.4 (iii), L is among the s edges of $\bar{\mathcal{C}}$ that intersect only in column j. Since $I \subseteq L - \{j\}$, there are $s - 1$ edges of $\bar{\mathcal{C}} \setminus I$ that intersect only in column j. Therefore, $|T| \leq s - 2$ implies $j \in T$. By symmetry among the vertices of $U - \{i\}$, it follows that $U - \{i\} \subseteq T$. So in particular $|T| \geq s - 1$, a contradiction. This proves Claim 1.

Suppose $\mathcal{C} \setminus I$ packs. Then it follows from Claim 1 that $\mathcal{C} \setminus I$ contains $s - 1$ disjoint edges $L_1, \ldots, L_{s-1}$.

Claim 2: None of $L_1, \ldots, L_{s-1}$ are edges of $\bar{\mathcal{C}}$.

Proof: Suppose that $L_1 \in E(\bar{\mathcal{C}})$ and let U_1 be its mate. Then $U_1 - (I \cup L_1)$ contains an edge T in $b(\mathcal{C})/(I \cup L_1)$. By assumption $q = rs - n + 1 \geq 3$. Thus

$$|T| \quad \leq \quad |U_1 - L_1| = |U_1| - q = s - q \leq s - 3$$

By Proposition 1.35, T is a transversal of $\mathcal{C} \setminus (I \cup L_1)$. But $L_2, \ldots, L_{s-1}$ are disjoint edges of $\mathcal{C} \setminus (I \cup L_1)$, which implies that every tranversal of $\mathcal{C} \setminus (I \cup L_1)$ has cardinality at least $s - 2$, a contradiction. This proves Claim 2.

By Corollary 4.5, the edges $L_1, \ldots, L_{s-1}$ have cardinality at least $r + 1$. Moreover they do not intersect I. Therefore we must have:

$$(r + 1)(s - 1) \leq n - |I| = rs - q + 1 - (r - q + 1) = rs - r$$

Thus $s \leq 1$, a contradiction. $\square$

4.2 Ideal Minimally Nonpacking Clutters

Minimally nonpacking clutters are either ideal or minimally nonideal. This follows from Theorem 1.8. Theorem 4.18 above discussed the minimally nonideal case. In this section, we discuss the ideal case. Q_6 is such an example, as seen in Exercise 1.9.

A clutter is *binary* if its edges have an odd intersection with its minimal transversals. Seymour [183] showed that Q_6 is the only ideal minimally nonpacking binary clutter. However, there are ideal minimally nonpacking clutters that are not binary, such as

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Note that, for this clutter, the minimum size of a transversal is 2. Other examples can be found in [64] but none is known with a minimum transversal of size greater than 2. Interestingly, all ideal minimally nonpacking clutters with a transversal of size 2 share strong structural properties with Q_6. A clutter $\mathcal{C}$ has the Q_6-*property* if $M(\mathcal{C})$ has 4 rows such that every column restricted to this set of rows contains two 0's and two 1's and each such 6 possible 0,1 vectors occurs at least once.

Theorem 4.20 (Cornuéjols, Guenin, Margot [64]) *Every ideal minimally nonpacking clutter with a transversal of size 2 has the Q_6-property.*

Conjecture 4.21 [64] *Every ideal minimally nonpacking clutter has a transversal of size 2.*

It is proved in [64] that this conjecture would imply Conjecture 1.6. Conjecture 1.6 can be reformulated in a form similar to Theorem 4.1.

Conjecture 4.22 (Conforti and Cornuéjols [41])
For a 0,1 matrix A, the following statements are equivalent:

(i) the matrix A has the MFMC property,

(ii) min $\{wx : Ax \geq \mathbf{1},\ x \geq 0\}$ has an integral optimal dual solution y for all $w \in \{0, 1, +\infty\}^n$.

Exercise 4.23 *Show that this conjecture is equivalent to Conjecture 1.6.*

Another equivalent conjecture is similar to Lovász's replication lemma (Lemma 3.3).

Conjecture 4.24 (Replication Conjecture) [41] *If $\mathcal{C}$ and all its minors pack, then for any $j \in V(\mathcal{C})$, the following clutter $\mathcal{C}_j$ packs.*

$$V(\mathcal{C}_j) = V(\mathcal{C}) \cup \{j'\}$$

$$E(\mathcal{C}_j) = E(\mathcal{C}) \cup \{A - \{j\} \cup \{j'\} : \ A \in E(\mathcal{C}) \text{ and } A \ni j\}.$$

Remark 4.25 *$\mathcal{C}_j$ packs if and only if* $\min\{wx : \ M(\mathcal{C})x \geq \mathbf{1}, \ x \in \{0,1\}^n\} = \max\{y\mathbf{1} : yM(\mathcal{C}) \leq w, \ y \in Z_+^m\}$, *for the vector w with $w_j = 2$ and $w_i = 1$ for $i \neq j$.*

Exercise 4.26 *Prove Remark 4.25.*

Exercise 4.27 *Show that Conjecture 4.24 is equivalent to Conjecture 4.22.*

4.3 Clutters such that $\tau_2(\mathcal{C}) < \tau_1(\mathcal{C})$

Let $\mathcal{C}$ be a clutter and let $M = M(\mathcal{C})$ be the associated 0,1 matrix. Let k be a positive integer and let

$$\tau_k(\mathcal{C}) = \min\{\mathbf{1}x : x \geq 0, \ Mx \geq \mathbf{1}, \ kx \ \textit{integral}\} \tag{4.3}$$

$$\nu_k(\mathcal{C}) = \max\{y\mathbf{1} : y \geq 0, \ yM \leq \mathbf{1}, \ ky \ \textit{integral}\}. \tag{4.4}$$

Clearly, $\mathcal{C}$ packs if and only if $\tau_1(\mathcal{C}) = \nu_1(\mathcal{C})$ and, for any clutter $\mathcal{C}$, there exists k large enough such that $\tau_k(\mathcal{C}) = \nu_k(\mathcal{C})$, since the LP's (4.3) and (4.4) have rational optimum solutions.

Theorem 4.28 (Ding [76]) *If $\mathcal{C}$ is a minor-minimal clutter such that $\tau_2(\mathcal{C}) < \tau_1(\mathcal{C})$, then either*
(i) $\mathcal{C}$ has a $\mathcal{J}_k$ minor, for $k \geq 2$, or
(ii) $\text{core}(\mathcal{C}) = \mathcal{C}^2_{2k-1}$, *for $k \geq 2$.*

A clutter is *diadic* if $|A \cap B| \leq 2$ for all $A \in E(\mathcal{C})$ and $B \in E(b(\mathcal{C}))$.

Theorem 4.29 (Ding [76]) *If $\mathcal{C}$ is a minor-minimal diadic clutter such that $\tau_2(\mathcal{C}) < \tau_1(\mathcal{C})$, then $\mathcal{C} = \mathcal{C}^2_{2k-1}$ or $b(\mathcal{C}^k_{2k-1})$, for $k \geq 2$.*

Conjecture 1.6 (and, equivalently, Conjectures 4.22 and 4.24) holds for diadic clutters.

Theorem 4.30 [64] *If $\mathcal{C}$ is a diadic clutter, then $\mathcal{C}$ has the MFMC property if and only if $\mathcal{C}$ has the packing property.*

Chapter 5

Odd Cycles in Graphs

In this chapter, we consider the clutter $\mathcal{C}$ of odd cycles in a graph G. Seymour [183] characterized exactly the graphs for which $\mathcal{C}$ has the MFMC property and Guenin [110] characterized exactly when $\mathcal{C}$ is ideal.

For edge weights $w \in R_+^{E(G)}$, consider the minimization problem (1.1). Recall that an integral solution to (1.1) is the incidence vector of a transversal T of $\mathcal{C}$. Since T intersects all odd cycles, $E(G) - T$ induces a bipartite graph. Therefore, a minimal transversal T of $\mathcal{C}$ is the complement of a cut $(W, \bar{W})$. In particular, when $\mathcal{C}$ is ideal, (1.1) finds a cut of maximum weight in G, i.e. (1.1) solves the famous *max cut* problem.

5.1 Planar Graphs

Orlova and Dorfman [152] showed that the clutter $\mathcal{C}$ of odd cycles is ideal when G is planar.

Theorem 5.1 (Orlova and Dorfman [152]) *In a planar graph, the clutter of odd cycles is ideal.*

Proof: Let G be a planar graph and D its dual. The bounded faces of G form a cycle basis. Thus any odd cycle of G is a symmetric difference of faces, an odd number of which are odd faces. Faces of G correspond to nodes of D. Let T be the set of odd degree nodes of D. An odd cycle of G corresponds to an edge set of D of the form $(W, \bar{W})$ where $W \cap T$ has odd cardinality, i.e. a T-cut of D. The clutter of T-cuts in D is ideal by the Edmonds-Johnson theorem (Theorem 2.1) and therefore so is the clutter of odd cycles in G. □

When $G = K_5$, the complete graph on 5 nodes, the clutter $\mathcal{C}$ of odd cycles is not ideal since $x_j = \frac{1}{3}$ for $j = 1, \ldots, 10$ is a fractional extreme point of the polyhedron $\{x \in R^{10} : \ M(\mathcal{C})x \geq \mathbf{1}\}$ (we leave this as an exercise).

Exercise 5.2 *Show that, when $G = K_5$, the clutter of odd cycles is not ideal.*

Barahona [3] observed that Theorem 5.1 has the following generalization.

Theorem 5.3 *In a graph not contractible to K_5, the clutter of odd cycles is ideal.*

This follows from a famous theorem of Wagner [207] stating that any edge-maximal graph not contractible to K_5 can be constructed recursively by pasting plane triangulations and copies of V_8 along K_3's and K_2's, where V_8 is the cycle $v_1, v_2, \ldots, v_8, v_1$ with chords $v_i v_{i+4}$ for $i = 1, 2, 3, 4$ (see Diestel [73] Theorem 8.3.4).

Exercise 5.4 *Prove that the odd cycle clutter of V_8 is ideal.*

Is there a converse to Barahona's theorem? In particular, is it true that, if the clutter of odd cycles is ideal in a graph G, then G is not contractible to K_5? The answer to the second question is no. For example, insert a node of degree 2 on every edge of K_5. The graph is now bipartite and the clutter of odd cycles has become the trivial clutter, which is ideal! The difficulty is that contraction of an edge changes odd cycles into even cycles and vice versa. To get a converse to Barahona's theorem, one needs to redefine contraction appropriately. It is convenient to work in the more general context of signed graphs.

5.2 Signed Graphs

Consider a graph G and a subset S of its edges. The pair (G, S) is called a *signed graph*, and S is called the *signature* of G. The edges in S are called *odd edges*. We say that a subset of edges of G is *odd* (resp. *even*) if it contains an odd (resp. even) number of edges in S. In particular we will talk about odd cycles of (G, S).

Consider a signed graph (G, S) and let C be any cut of G. Since C intersects every cycle with even parity, it follows that (G, S) and $(G, S \triangle C)$ have the same set of odd cycles (where $\triangle$ denotes the symmetric difference of two sets). We call the operation which consists of replacing S by $S \triangle C$ a *signature-exchange*.

In a signed graph (G, S), *deleting* an edge means removing it from the graph. *Contracting* an edge e means first doing a signature-exchange if necessary so that the edge e is even, i.e. $e \notin S$, and then removing the edge and identifying its endnodes.

Let C and D be disjoint edge sets. One can readily verify that all the signed graphs obtained by deleting the edges in D and contracting the edges in C are identical (up to signature-exchanges), no matter in which order the contractions and deletions are performed. A signed graph obtained from (G, S) by a sequence of contractions and deletions and signature-exchanges is called a *minor* of (G, S).

Exercise 5.5 *Let $\mathcal{C}$ denote the clutter of odd cycles in a signed graph (G, S). Show that every minor of $\mathcal{C}$ is the clutter of odd cycles in a signed graph (G', S') obtained as a minor of (G, S).*

A signed complete graph K_5 on five nodes is called an *odd*-K_5 if all its edges are odd. Recently, Guenin proved the following theorem.

Theorem 5.6 (Guenin [110]) *The clutter of odd cycles of a signed graph (G, S) is ideal if and only if (G, S) has no odd-K_5 minor.*

A clutter is *binary* (see Section 5.4) if its edges and its minimal transversals intersect in an odd number of vertices. The clutter of odd cycles in a signed graph is a binary clutter. Theorem 5.6 is a special case of a famous conjecture of Seymour [183], [187] (Conjecture 5.26) on ideal binary clutters. In [183], Seymour characterized the binary clutters that have the MFMC property. Specialized to the clutter of odd cycles, this theorem is the following.

Theorem 5.7 (Seymour [183]) *The clutter of odd cycles of a signed graph (G, S) has the MFMC property if and only if (G, S) has no odd-K_4 minor.*

Exercise 5.8 *Prove the following direction of Theorem 5.7: If the clutter of odd cycles has the MFMC property in a signed graph G, then G has no odd-K_4 minor.*

Exercise 5.9 *Prove the following direction of Theorem 5.6: If the clutter of odd cycles is ideal in a signed graph G, then G has no odd-K_5 minor.*

5.3 Proof Outline of Guenin's Theorem

One direction of Guenin's theorem is easy (Exercise 5.9). For the converse we need two lemmas on mni binary clutters. Observe at the outset that $\mathcal{J}_t$ is not binary. Denote the core of a mni clutter $\mathcal{A}$ by $\bar{\mathcal{A}}$.

Lemma 5.10 *Let $\mathcal{A}$ be a mni binary clutter and $C_1, C_2 \in E(\bar{\mathcal{A}})$. If $C \subseteq C_1 \cup C_2$ and $C \in E(\mathcal{A})$ then either $C = C_1$ or $C = C_2$.*

Proof: Let r denote the cardinality of the edges in $E(\bar{\mathcal{A}})$.

Case 1 $|C| = r$.

It follows from Corollary 4.5 that $C \in E(\bar{\mathcal{A}})$. Let U be the mate of C and $d = |C \cap U| \geq 2$. Since $\mathcal{A}$ is binary, d must be odd. So in particular $d \geq 3$. Since $C \subseteq C_1 \cup C_2$, we must have $|U \cap C_1| > 1$ or $|U \cap C_2| > 1$. This implies that U is the mate of C_1 or C_2, i.e. that $C = C_1$ or $C = C_2$.

Case 2 $|C| > r$.

Let $T = C \triangle C_1 \triangle C_2$. Since every minimal transversal U has an odd intersection with C, C_1 and C_2, we have $T \cap U \neq \emptyset$. Therefore T contains an odd cycle. Now

$$|T| = |C \triangle C_1 \triangle C_2| \leq |C_1| + |C_2| - |C| < r,$$

a contradiction. □

For example, in an odd-K_5, Lemma 5.10 says that if an odd cycle C has edges contained in the union of two triangles, then C is one of these two triangles.

Lemma 5.11 *Let $\mathcal{A}$ be a mni binary clutter and $\mathcal{B}$ its blocker. For any $e \in V(\mathcal{A})$ there exist $C_1, C_2, C_3 \in E(\bar{\mathcal{A}})$ and $U_1, U_2, U_3 \in E(\bar{\mathcal{B}})$ such that*

(i) $C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3 = \{e\}$

(ii) $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{e\}$

(iii) $C_i \cap U_j = \{e\}$ *if* $i \neq j$ *and* $|C_i \cap U_j| = d \geq 3$ *if* $i = j$.

(iv) For all $e_i \in U_i$ and $e_j \in U_j$, there exists $C \in E(\mathcal{A})$ with $C \cap U_i = \{e_i\}$ and $C \cap U_j = \{e_j\}$.

Proof: By Theorem 4.4(iii) there exist s edges $C_1, \dots, C_s \in E(\mathcal{A})$ such that $C_1 - \{e\}, \dots, C_s - \{e\}$ are disjoint. Moreover, exactly $d = rs - n + 1 \geq 2$ of these edges, say $C_1, \dots, C_d$, contain vertex e. As $\mathcal{A}$ is binary, $d \geq 3$. This proves (i). Let U_1, U_2, U_3 be the mates of C_1, C_2 and C_3. Note that (iii) is immediate. (ii) can be derived from Theorem 4.3 and Theorem 4.4(iii) (we omit the proof). Let us prove (iv). Let $T = U_i \cup U_j - \{e_i, e_j\}$. Since $\mathcal{A}$ is binary, so is its blocker $\mathcal{B}$. By Lemma 5.10, there is no $U \in E(\mathcal{B})$ with $U \subseteq T$. Thus $V(\mathcal{A}) - T$ intersects every edge of $\mathcal{B}$. Since $b(\mathcal{B}) = \mathcal{A}$ (Theorem 1.15), it follows that $V(\mathcal{A}) - T$ contains an edge C of $\mathcal{A}$. But $C \cap U_i \neq \emptyset$ and $C \cap U_j \neq \emptyset$. Thus, by construction, $C \cap U_i = \{e_i\}$ and $C \cap U_j = \{e_j\}$. □

Let $\mathcal{A}$ be the clutter of odd cycles of a signed graph (G, S) and assume that $\mathcal{A}$ is *mni*. To prove Theorem 5.6, it suffices to show that (G, S) is equal (up to signature-exchanges) to an odd-K_5. Let $\mathcal{B} = b(\mathcal{A})$. Recall that $\mathcal{A}$ and $\mathcal{B}$ are binary. Let e be any edge of G, and let C_1, C_2, C_3 be the odd cycles of (G, S) given in Lemma 5.11. We know that these cycles intersect exactly in edge e. Let w, w' be the endnodes of e.

Lemma 5.12 *The only nodes common to more than one of C_1, C_2, C_3 are the endnodes of e.*

Proof: Suppose, for instance, that C_1 and C_2 have a node t in common distinct from w and w'. Let P (resp. P') be the path in $C_1 - e$ from w (resp. w') to t. Let Q (resp. Q') be the path in $C_2 - e$ from w (resp. w') to t. Because of signature-exchanges, we may assume that e is odd and that paths P, P' are both even. If Q is odd, then $P \cup Q$ contains an odd cycle. If Q is even, then $P' \cup Q \cup \{e\}$ contains an odd cycle. Both cases contradict Lemma 5.10. □

Lemma 5.12 implies that we may assume (after a sequence of signature-exchanges) that e is the only odd edge in C_1, C_2 and C_3. Let U_1, U_2, U_3 be the sets of edges of G given in Lemma 5.11. The cycle C_1 has at least two edges, distinct from e, in common with U_1. Choose such an edge $e_1 \in (C_1 \cap U_1) - e$ with endnodes t_1, t_1' such that the path P included in $C_1 - e$ from w to t_1 contains exactly one edge of U_1 namely e_1 (possibly relabeling the endpoints of e_1). Similarly, we define edges e_2, e_3 and nodes t_2, t_3 for C_2, C_3. Note that t_1, t_2, t_3 are distinct from the endnodes w, w' of e.

Lemma 5.13 *There are odd paths P_{ij} with endnodes t_i, t_j for each $i, j \in \{1, 2, 3\}$ and $i \neq j$.*

Proof: By Lemma 5.11(iv), there is a an odd cycle C with $C \cap U_1 = \{e_1\}, C \cap U_2 = \{e_2\}$ and $e \notin C$. The cycle C can be written as $\{e_1, e_2\} \cup Q \cup Q'$ where Q and Q' are paths disjoint from U_1 and U_2. Since C is odd and all edges in $C_1 \cup C_2 \cup C_3 - e$ are even, exactly one of Q or Q' is odd (say Q is odd). We leave it as an exercice to check that if the endnodes of Q are not t_1 and t_2 then (G, S) contains an odd cycle C' disjoint from either U_1 or U_2. But this is a contradiction as $C \in E(\mathcal{A})$ and $U_1, U_2 \in E(\mathcal{B})$. □

Exercise 5.14 *Complete the proof of Lemma 5.13.*

Suppose all internal nodes of the paths P_{12}, P_{13}, P_{23} have degree two. Then do a signature-exchange using the cut where one of the shores consists of w, w'. By a sequence of contractions, replace every odd path, with internal nodes of degree two, by a single odd edge. The resulting graph is an odd-K_5. The hardest part of the proof deals with the case where P_{12}, P_{23} and P_{31} are not node disjoint. See [110] for the proof in this case.

5.4 Binary Clutters

A clutter is *binary* if its edges and its minimal transversals intersect in an odd number of vertices. It follows from the definition that a clutter is binary if and only if its blocker is binary. An equivalent formulation is given by Lehman.

Proposition 5.15 (Lehman [131], see also Seymour [181]) *A clutter $\mathcal{C}$ is binary if and only if, for any three edges S_1, S_2, S_3 of $\mathcal{C}$, the set $S_1 \triangle S_2 \triangle S_3$ contains an edge of $\mathcal{C}$.*

Proof: Let $\mathcal{C}$ be a binary clutter and $S = S_1 \triangle S_2 \triangle S_3$ where $S_1, S_2, S_3 \in E(\mathcal{C})$. Since every minimal transversal T has an odd intersection with S_1, S_2 and S_3, we have $S \cap T \neq \emptyset$. Therefore S contains an edge of $\mathcal{C}$.

Conversely, assume that for any three edges S_1, S_2, S_3 of $\mathcal{C}$, the set $S_1 \triangle S_2 \triangle S_3$ contains an edge of $\mathcal{C}$. We leave it as an exercise to show that, for any odd number of edges $S_1, \ldots, S_k$ of $\mathcal{C}$, the set $S_1 \triangle \ldots \triangle S_k$ contains an edge of $\mathcal{C}$. Now consider any $S \in E(\mathcal{C})$, $T \in E(b(\mathcal{C}))$ and let $S \cap T = \{x_1, \ldots, x_k\}$. Since $T - x_i$ is not a transversal of $\mathcal{C}$, there exists an edge S_i of $\mathcal{C}$ such that $T \cap S_i = \{x_i\}$. It follows that $T \cap (S \triangle S_1 \triangle \ldots \triangle S_k) = \emptyset$. Therefore $S \triangle S_1 \triangle \ldots \triangle S_k$ does not contain an edge of $\mathcal{C}$. It follows that k is odd. $\Box$

Exercise 5.16 *Let $\mathcal{C}$ be a clutter such that, for any three edges S_1, S_2, S_3, the set $S_1 \triangle S_2 \triangle S_3$ contains an edge of $\mathcal{C}$. Show that, for any odd number of edges $S_1, \ldots, S_k$ of $\mathcal{C}$, the set $S_1 \triangle \ldots \triangle S_k$ contains an edge of $\mathcal{C}$.*

Exercise 5.17 *Show that, in a signed graph, the clutter of odd cycles is binary.*

Let $\mathcal{P}_4$ be the clutter with four vertices and the following three edges: $E(\mathcal{P}_4) = \{\{1,2\},\{2,3\},\{3,4\}\}$.

Exercise 5.18 *Show that neither $\mathcal{P}_4$ nor $\mathcal{J}_t$ is a binary clutter, for $t \geq 2$.*

Theorem 5.19 (Seymour [181]) *$\mathcal{C}$ is a binary clutter if and only if $\mathcal{C}$ has no minor $\mathcal{P}_4$ or $\mathcal{J}_t$, for $t \geq 2$.*

The following clutters (and their blockers!) are examples of binary clutters.

Example 5.20 *The clutter of st-paths in a graph.*

Example 5.21 *The clutter of two-commodity paths in a graph.*

Example 5.22 *The clutter of T-joins in a graft (G, T).*

Example 5.23 *The clutter of odd cycles in a signed graph.*

We present two more examples.

st-T-Cuts

Recently, Goemans and Ramakrishnan [105] introduced a generalization of st-cuts, T-cuts and two-commodity cuts as follows. In a graph G, let s, t be two distinct nodes and let T be a node set of even cardinality. An *st-T-cut* is a T-cut $(W, \bar{W})$ where W contains exactly one of s or t. The st-cut clutter is obtained when $T = \{s, t\}$, the T-cut clutter is obtained when t is an isolated node and the two-commodity cut clutter is obtained when $T = \{s', t'\}$.

Exercise 5.24 *Show that the clutter of st-T-cuts is binary.*

Odd st-Walks

Guenin [112] considers the following generalization of the odd cycle clutter. Let (G, S) be a signed graph and let s, t be two nodes of G. A subset of edges of G is an *odd st-walk* if it is an odd st-path or the union of an even st-path P and an odd cycle C where P and C share at most one node. The odd cycle clutter is obtained when $s = t$.

Exercise 5.25 *Show that the clutter of odd st-walks is binary.*

5.4.1 Seymour's Conjecture

Recall that $\mathcal{F}_7$ denotes the clutter with 7 vertices and 7 edges corresponding to points and lines of the Fano plane (finite projective geometry on 7 points). It is easy to verify that $\mathcal{F}_7$ is binary, *mni* and that $b(\mathcal{F}_7) = \mathcal{F}_7$ (see Exercises 4.15 and 5.27).

Let K_5 denote the complete graph on five nodes. We let $\mathcal{O}_{K_5}$ denote the binary clutter whose vertices are the edges of K_5 and whose edges are the odd cycles of K_5. So $\mathcal{O}_{K_5}$ has 10 edges of cardinality three and 12 edges of cardinality five. $\mathcal{O}_{K_5}$ is binary and *mni*. It follows that $b(\mathcal{O}_{K_5})$ is binary and *mni* (see Exercises 4.16 and 5.27).

Conjecture 5.26 (Seymour [183]) *A binary clutter is ideal if and only if it contains no $\mathcal{F}_7$, $\mathcal{O}_{K_5}$ or $b(\mathcal{O}_{K_5})$ minor.*

Exercise 5.27 *Show that $\mathcal{F}_7$ and $\mathcal{O}_{K_5}$ are binary clutters.*

Theorems 5.6 is a special case of Seymour's conjecture since one can verify that neither $\mathcal{F}_7$ nor $b(\mathcal{O}_{K_5})$ is an odd cycle clutter of a signed graph.

Similarly Theorems 1.22 and 2.1 are special cases of Seymour's conjecture since $\mathcal{F}_7$, $\mathcal{O}_{K_5}$ and $b(\mathcal{O}_{K_5})$ are neither T-join nor two-commodity cut clutters.

One can also check that $\mathcal{O}_{K_5}$ is not an st-T-cut clutter and that $b(\mathcal{O}_{K_5})$ is not an odd-st-walk clutter. Recently, Guenin proved two more cases of Seymour's conjecture.

Theorem 5.28 (Guenin [112]) *A clutter of odd st-walks is ideal if and only if it has no $\mathcal{F}_7$ or $\mathcal{O}_{K_5}$ minor.*

Theorem 5.29 (Guenin [112]) *A clutter of st-T-cuts is ideal if and only if it has no $\mathcal{F}_7$ or $b(\mathcal{O}_{K_5})$ minor.*

Theorem 5.28 implies Theorem 5.6 and Theorem 5.29 implies Theorems 1.22 and 2.1.

5.4.2 Seymour's MFMC Theorem

Denote by $\mathcal{Q}_6$ the clutter with six vertices and the following four edges:

$$E(\mathcal{Q}_6) = \{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}.$$

$\mathcal{Q}_6$ is the clutter of triangles of K_4. It is easy to check that $\mathcal{Q}_6$ is a binary clutter but does not have the MFMC property (Exercises 5.17 and 1.9).

Theorem 5.30 (Seymour [183]) *A binary clutter has the MFMC property if and only if it does not have a $\mathcal{Q}_6$ minor.*

Specializing Seymour's max-flow min-cut theorem (Theorem 5.30) to the above examples of binary clutters, we get graph theoretic results. Theorems 2.11, 2.16 and 5.7 are three such results that have been mentioned already.

Chapter 6

$0, \pm 1$ Matrices and Integral Polyhedra

The concepts of perfect and of ideal 0,1 matrices can be extended to $0, \pm 1$ matrices. Given a $0, \pm 1$ matrix A, denote by $n(A)$ the column vector whose i^{th} component is the number of -1's in the i^{th} row of matrix A. A $0, \pm 1$ matrix A is *perfect* if the polytope $\{x : \; Ax \leq \mathbf{1} - n(A), \; 0 \leq x \leq \mathbf{1}\}$ is integral. Similarly, a $0, \pm 1$ matrix A is *ideal* if the polytope $\{x : \; Ax \geq \mathbf{1} - n(A), \; 0 \leq x \leq \mathbf{1}\}$ is integral. A matrix is *totally unimodular* if every square submatrix has determinant equal to $0, \pm 1$. In particular, all entries are $0, \pm 1$. A milestone result in the study of integral polyhedra, due to Hoffman and Kruskal [122], is that the following statements are equivalent for an integral matrix A.

- The polyhedron $\{x \geq 0 : \; Ax \leq b\}$ is integral for each integral vector b,
- A is totally unimodular.

We prove this in Section 6.1. It follows from this result that a totally unimodular matrix is both perfect and ideal.

A $0, \pm 1$ matrix is *balanced* if, in every square submatrix with exactly two nonzero entries per row and per column, the sum of the entries is a multiple of four. The class of balanced $0, \pm 1$ matrices properly includes totally unimodular $0, \pm 1$ matrices. See Figure 6.1.

In Section 6.2, we will prove the following equivalence.

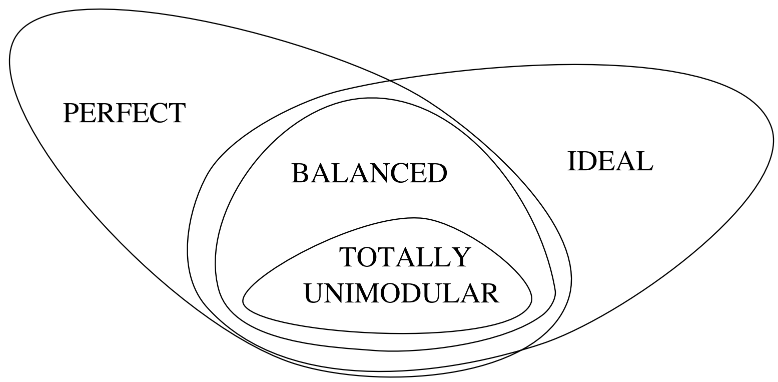

Figure 6.1: Classes of $0, \pm 1$ matrices.

Theorem 6.1 (Conforti and Cornuéjols [42])

Let A be a $0, \pm 1$ matrix. Then the following statements are equivalent.

(i) the matrix A is balanced,

(ii) every submatrix of A is perfect,

(iii) every submatrix of A is ideal.

It is well known that several problems in propositional logic, such as SAT, MAXSAT and logical inference, can be written as integer programs of the form

$$min\{cx : \; Ax \geq \mathbf{1} - n(A), \; x \in \{0,1\}^n\}.$$

These problems are NP-hard in general but they can be solved in polytime by linear programming when the corresponding $0, \pm 1$ matrix A is ideal. In fact, in this case, SAT and logical inference can be solved very fast by unit resolution [43]. This is discussed in Section 6.3.

6.1 Totally Unimodular Matrices

A matrix is *totally unimodular* if all its square submatrices have a determinant equal to 0, +1 or -1. In particular, all entries of a totally unimodular matrix must be equal to 0, +1 or -1. The following result shows that this concept is important in integer programming. We follow the proof of Veinott and Dantzig [204]

Theorem 6.2 (Hoffman and Kruskal [122]) *A* $0, \pm 1$ *matrix* A *is totally unimodular if and only if* $\{x \in R^n_+ : Ax \leq b\}$ *is an integral polyhedron for every integral vector* b.

Proof: First, assume that A is totally unimodular and let $\bar{x}$ be an extreme point of $\{x \in R^n_+ : Ax \leq b\}$ for some integral vector b. Then $\bar{x}$ is the solution of a linear system of n equations $Dx = d$, where D is nonsingular, taken from the inequalities $x \geq 0, Ax \leq b$ that are tight at $\bar{x}$. Since A is totally unimodular and b is integral, it follows from Cramer's rule that $\bar{x}$ is integral.

Conversely, let A be an $m \times n$ integral matrix and assume that $\{x \in R^n_+ : Ax \leq b\}$ has integral extreme points for every integral vector b. To prove that A is totally unimodular, we consider a nonsingular square submatrix A' of A, we complete it into an $m \times m$ nonsingular submatrix B of (A, I_m) by using the columns of I_m that have 0's in the rows of A', and we show that det $B = \pm 1$. Let β^j be the jth column of B^{-1}, let y be an integral vector such that $y + \beta^j \geq 0$ and let $z = y + \beta^j$. Then $Bz = By + e^j$ where e^j denotes the jth unit vector. So, the point $\bar{x}$ defined by $\bar{x}_j = z_j$ when j is a column of B and $\bar{x}_j = 0$ otherwise is an extreme point of $\{x \in R^n_+ : Ax \leq b\}$ for $b = By + e^j$. By assumption, it follows that z is integral. So $\beta^j = z - y$ is integral and therefore B^{-1} is an integral matrix. Since B is also an integral matrix and (det B)(det B^{-1}) $= 1$, it follows that det $B = \pm 1$. □

Several elegant characterizations of total unimodularity are known. We present those of Ghouila-Houri [104] and Camion [26]. We follow Padberg's presentation of the proofs [156]. A $0, \pm 1$ matrix A has an *equitable bicoloring* if its columns can be partitioned into blue columns and red columns so that, for every row of A, the sum of the entries in the blue columns differs from the sum of the entries in the red columns by at most one.

Theorem 6.3 (Ghouila-Houri [104]) *A* $0, \pm 1$ *matrix* A *is totally unimodular if and only if every submatrix of* A *has an equitable bicoloring.*

Proof: Assume A is totally unimodular and, for any submatrix B of A, define the polytope

$$P = \{x : \ \ 0 \leq x \leq \mathbf{1}, \ \ \lfloor \frac{1}{2} B\mathbf{1} \rfloor \leq Bx \leq \lceil \frac{1}{2} B\mathbf{1} \rceil \}.$$

The definition of total unimodularity implies that $(B, -B, I)$ is totally unimodular. So P is an integral polytope by the Hoffman-Kruskal theorem (Theorem 6.2). Since P is nonempty (indeed $(\frac{1}{2}, \ldots, \frac{1}{2}) \in P$), it follows that P contains a 0,1 point $\bar{x}$. Now, an equitable bicoloring is obtained by coloring red the columns j where $\bar{x}_j = 1$ and blue those where $\bar{x}_j = 0$.

Now we prove the converse. Assume that every submatrix of A has an equitable bicoloring. To show that every $k \times k$ submatrix of A has a determinant equal to $0, \pm 1$, we use induction on k. The result holds for $k = 1$. Suppose it is true for $k \geq 1$ and let B be a *nonsingular* $(k+1) \times (k+1)$ submatrix of A. Let $d = \det B$. Expressing the entries of B^{-1} using cofactors, it follows from the induction hypothesis that each entry of the matrix dB^{-1} equals $0, \pm 1$. Let β be the first column of dB^{-1} and let $q = B|\beta|$. Note that q_i is even for $i = 2, \ldots, k+1$. Suppose that q_1 is also even. Then the equitable bicoloring assumption, applied to the column submatrix of B with columns j such that $\beta_j \neq 0$, implies that there exists a $0, \pm 1$ vector $y \neq 0$ such that $By = 0$, contradicting the nonsingularity of B. So q_1 is odd. Now the equitable bicoloring assumption implies that there exists a $0, \pm 1$ vector y such that $By = e^1$, the first unit vector. Since $B\beta = de^1$, it follows that $\beta = dy$. Now, since $\beta \neq 0$ has $0, \pm 1$ entries, it follows that $d = \pm 1$. □

Exercise 6.4 *Let A be a totally unimodular matrix. Show that, if A has an even number of nonzero entries in each row and column, then the sum of the entries of A is a multiple of 4. [Hint: Use Theorem 6.3.]*

Theorem 6.5 (Camion [26]) *A $0, \pm 1$ matrix A is totally unimodular if and only if, in every submatrix with an even number of nonzero entries per row and per column, the sum of the entries is a multiple of four.*

Proof: This result follows from Exercise 6.4 and the next theorem. □

A $0, \pm 1$ matrix A is *minimally non-totally unimodular* if it is not totally unimodular, but every proper submatrix has that property. Clearly, if a $0, \pm 1$ matrix is not totally unimodular, then it contains a minimally non-totally unimodular submatrix.

Theorem 6.6 (Camion [27] and Gomory (cited in [27])) *Let A be a $0, \pm 1$ minimally non-totally unimodular matrix. Then A is square, $det(A) = \pm 2$, and A^{-1} has only $\pm \frac{1}{2}$ entries. Furthermore, each row and each column of A has an even number of nonzeroes and the sum of all entries in A equals 2 modulo 4.*

Proof: Clearly A is square, say $n \times n$. If $n = 2$, then indeed, $\det A = \pm 2$. Now assume $n \geq 3$. Since A is nonsingular, it contains an $(n-2) \times (n-2)$ nonsingular submatrix B. Let $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ and $U = \begin{pmatrix} B^{-1} & 0 \\ -DB^{-1} & I \end{pmatrix}$. Then $\det U = \pm 1$ and $UA = \begin{pmatrix} I & B^{-1}C \\ 0 & E - DB^{-1}C \end{pmatrix}$. We claim that the 2×2 matrix $E - DB^{-1}C$ has all entries equal to $0, \pm 1$. Suppose to the contrary that $E - DB^{-1}C$ has an entry different from $0, \pm 1$ in row i and column j. Denoting the corresponding entry of E by e_{ij}, the corresponding column of C by c^j and row of D by d^i,

$$\begin{pmatrix} B^{-1} & 0 \\ -d^iB^{-1} & 1 \end{pmatrix} \begin{pmatrix} B & c^j \\ d^i & e_{ij} \end{pmatrix} = \begin{pmatrix} I & B^{-1}c^j \\ 0 & e_{ij} - d^iB^{-1}c^j \end{pmatrix}$$

and consequently A has an $(n-1) \times (n-1)$ submatrix with a determinant different from $0, \pm 1$, a contradiction.

Consequently, $\det A = \pm \det UA = \pm \det(E - DB^{-1}C) = \pm 2$.

So, every entry of A^{-1} is equal to $0, \pm \frac{1}{2}$. If α denotes a column of A^{-1} with at least one 0 component, then, applying Ghouila-Houri's theorem to the column submatrix of A with columns j such that $\alpha_j \neq 0$, it follows from $A|\alpha| \equiv 0 \pmod 2$ that there exists a $0, \pm 1$ vector $y \neq 0$ such that $Ay = 0$, contradicting the nonsingularity of A. So A^{-1} has only $\pm \frac{1}{2}$ entries.

This property and the fact that AA^{-1} and $A^{-1}A$ are integral, imply that A has an even number of nonzero entries in each row and column.

Finally, let α denote a column of A^{-1} and $S = \{i : \alpha_i = +\frac{1}{2}\}$ and $\bar{S} = \{i : \alpha_i = -\frac{1}{2}\}$. Let k denote the sum of all entries in the columns of A indexed by $\bar{S}$. Since $A\alpha$ is a unit vector, the sum of all entries in the columns of A indexed by S equals $k+2$. Since every column of A has an even number of nonzero entries, k is even, say $k = 2p$ for some integer p. Therefore, the sum of all entries in A equals $4p+2$. □

In Figure 6.2, we give the bipartite representations of four minimally non totally unimodular 0,1 matrices.

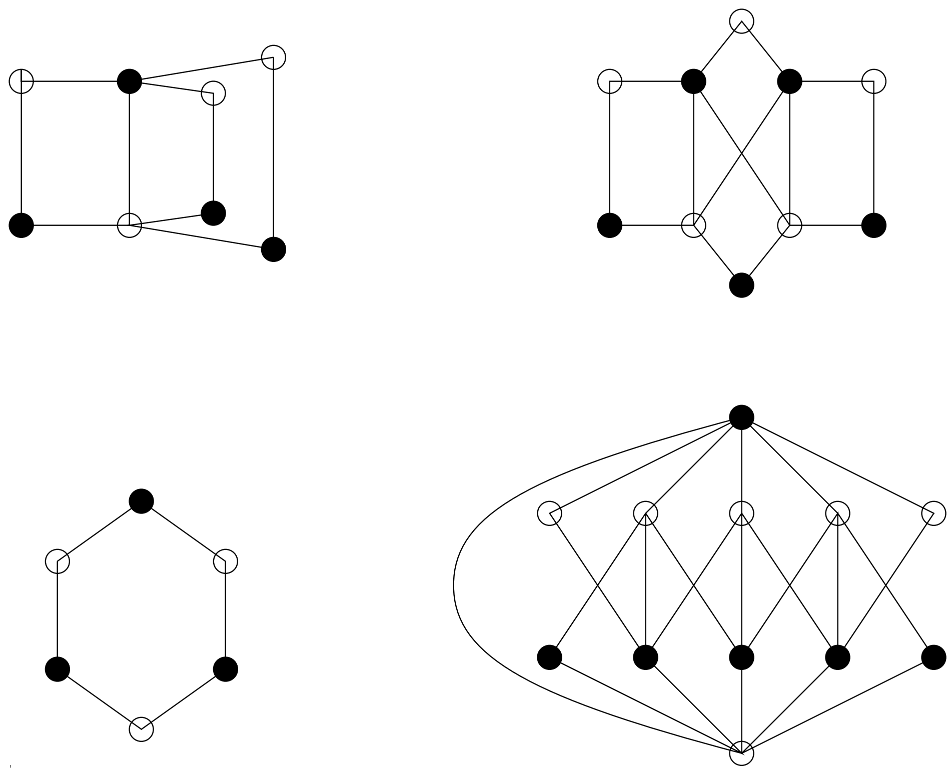

Figure 6.2: Bipartite representations of four minimally non totally unimodular 0,1 matrices

Exercise 6.7 *Show that a $0, \pm 1$ matrix A is totally unimodular if and only if all its square submatrices with an even number of nonzero entries in each row and column are singular.*

Exercise 6.8 *Let A be a $0, \pm 1$ matrix with exactly two nonzero entries in each row and column, such that no proper submatrix of A has this property. Show that A is singular if and only if the sum of its entries is a multiple of 4.*

A $0, \pm 1$ matrix with exactly two nonzero entries in each row and column, such that no proper submatrix has this property, is called a *hole matrix*. It is a *balanced hole matrix* if the sum of the entries is a multiple of 4 and it is *unbalanced* otherwise. Minimally non-totally unimodular matrices are either unbalanced hole matrices or they are balanced $0, \pm 1$ matrices. Padberg [156] conjectured that, for every minimally non-totally unimodular matrix A, there exists a linear transformation R that preserves total unimodularity such that RA is an unbalanced hole matrix. This conjecture is not correct (see [66]), but the following related question is open. Is it true that, for any minimally non-totally unimodular matrix A, there exists a totally unimodular matrix R such that RA is an unbalanced hole matrix?

By Theorem 6.6, the inverse of a minimally non-totally unimodular matrix is of the form

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & \mathbf{1}^T \\ \mathbf{1} & 2U - J \end{pmatrix}$$

up to multiplying rows and columns by -1, where U is a 0,1 matrix and J is the matrix filled with 1's. The matrix U has interesting properties. Truemper [191] showed that U is totally unimodular. Furthermore, several matrices derived from U must also be totally unimodular since A remains totally unimodular when multiplying rows or columns by -1. The *row i complement* of a 0,1 matrix B is derived from B by the following operation: for each column j such that $b_{ij} = 1$, replace b_{kj} by $1 - b_{kj}$ for each $k \neq i$. The *column j complement* operation is defined similarly. A 0,1 matrix B is *complement totally unimodular* if B and all matrices derivable from B by a sequence of row and column complement operations are totally unimodular. Truemper showed that the 0,1 matrix U in A^{-1} above is complement totally unimodular. Conversely, any nonsingular complement totally unimodular matrix U gives rise to a valid A^{-1}. This result is expressed in terms of A as follows.

Theorem 6.9 (Truemper [191]) *Let U be a square nonsingular complement totally unimodular matrix. Then the matrix*

$$A = \begin{pmatrix} \mathbf{1}^T U^{-1} \mathbf{1} - 2 & \mathbf{1}^T U^{-1} \\ U^{-1}\mathbf{1} & U^{-1} \end{pmatrix}$$

is a minimally non-totally unimodular matrix. Moreover, every $0, \pm 1$ minimally non-totally unimodular matrix A can be constructed this way, up to multiplying rows or columns by -1.

Exercise 6.10 *Prove Theorem 6.6 using Theorem 6.9.*

Truemper [191] (see also [194]) showed that a minimally non-totally unimodular matrix has a row or column that contains exactly two nonzero entries.

6.2 Balanced Matrices

A $0, 1$ matrix is *balanced* if it does not contain a square submatrix of odd order with two ones per row and per column. This notion was introduced by Berge [6].

Conjecture 6.11 (Conforti and Rao [57]) *If a 0,1 matrix is balanced, there exists a sequence in which its 1's can be turned into 0's, one at a time, so that all intermediate matrices are balanced.*

Berge [7] showed that, if A is balanced, then both the packing polytope $\{x \geq 0 : \; Ax \leq \mathbf{1}\}$ and the covering polyhedron $\{x \geq 0 : \; Ax \geq \mathbf{1}\}$ are integral. (The proof will be given, see Theorem 6.16 below.) It follows that, if A is balanced, the linear system $x \geq 0, \; Ax \leq \mathbf{1}$ is TDI (recall Corollary 3.12). Fulkerson, Hoffman and Oppenheim [92] also showed that, if A is balanced, then the linear system $x \geq 0, \; Ax \geq \mathbf{1}$ is TDI (see Theorem 6.17 below).

Balanced 0,1 matrices can also be viewed as a natural generalization of bipartite graphs. This is the motivation that led Berge to introduce the notion of balancedness. This point of view is developed in Section 6.2.3. Further results on balanced 0,1 matrices can be found in [9], [10], [11], [47], [69]. See [46] for a survey.

Several of these results can be extended to $0, \pm 1$ matrices. A $0, \pm 1$ matrix is *balanced* if, in every square submatrix with exactly two nonzero entries per row and per column, the sum of the entries is a multiple of 4. This notion was introduced by Truemper [192] and it generalizes that of balanced $0, 1$ matrix. Furthermore, every totally unimodular matrix is a balanced $0, \pm 1$ matrix. Using terminology introduced in the previous section, a $0, \pm 1$ matrix is balanced if and only if it does not contain an unbalanced hole submatrix.

6.2.1 Integral Polytopes

Given a $0, \pm 1$ matrix A, let $n(A)$ be the column vector whose i^{th} component is the number of -1's in the i^{th} row of matrix A. The *generalized set covering polytope* is $Q(A) = \{x \in R^n : \ Ax \geq \mathbf{1} - n(A), \ 0 \leq x \leq \mathbf{1}\}$. First, we consider the generalized set partitioning polytope $\{x \in R^n : Ax = \mathbf{1} - n(A), \ 0 \leq x \leq \mathbf{1}\}$.

Theorem 6.12 (Conforti and Cornuéjols [43]) *If A is a balanced $0, \pm 1$ matrix, then the polytope $P(A) = \{x \in R^n : \ Ax = \mathbf{1} - n(A), \ 0 \leq x \leq \mathbf{1}\}$ is an integral polytope.*

Proof: Assume that A contradicts the theorem and has the smallest size (number of rows plus number of columns). Then $P(A)$ is nonempty. Let $\bar{x}$ be a fractional extreme point of $P(A)$. By the minimality of A, $0 < \bar{x}_j < 1$ for all j. It follows that A is square and nonsingular.

Let $a^1, \ldots, a^n$ denote the row vectors of A and let A_i be the $(n-1) \times n$ submatrix of A obtained by removing row a^i. Then $\bar{x}$ belongs to a face of dimension 1 of the polytope $P(A_i) = \{x \in R^n : A_i x = \mathbf{1} - n(A_i), \ 0 \leq x \leq \mathbf{1}\}$. So,

$$\bar{x} = \lambda x^S + (1 - \lambda) x^T$$

where x^S, x^T are extreme points of $P(A_i)$. W.l.o.g. $a^i x^S < 1 - n_i(A)$ and $a^i x^T > 1 - n_i(A)$. By the minimality of A, all the extreme points of $P(A_i)$ have $0, 1$ components.

Let $S = \{j : \ x_j^S = 1\}$ and $T = \{j : \ x_j^T = 1\}$. Since $0 < \bar{x}_j < 1$ for all j, $S \cap T = \emptyset$ and $|S \cup T| = n$. Let k be any row of A_i. Since both x^S and x^T satisfy $a^k x = 1 - n(a^k)$, it follows that row k contains exactly two nonzero entries. Applying this argument to two different matrices A_i, it follows that every row of A contains exactly two nonzero entries.

If A has a column j with only one nonzero entry a_{kj}, remove column j and row k. Since A is nonsingular, the resulting matrix is also nonsingular. Repeating this process, we get a square nonsingular matrix B of order at least 2, with exactly two nonzero entries in each row and column. Exercise 6.8, which is routine, implies that B is not balanced. Therefore A is not balanced. $\Box$

Theorem 6.13 *Let A be a balanced $0, \pm 1$ matrix with rows $a^i, i \in S$, and let S_1, S_2, S_3 be a partition of S. Then*

$$\begin{aligned} R(A) = \{x \in R^n : \quad & a^i x \geq 1 - n(a^i) \quad for\ i \in S_1, \\ & a^i x = 1 - n(a^i) \quad for\ i \in S_2, \\ & a^i x \leq 1 - n(a^i) \quad for\ i \in S_3, \\ & 0 \leq x \leq \mathbf{1}\} \end{aligned}$$

is an integral polytope.

Proof: If $\bar{x}$ is an extreme point of $R(A)$, it is an extreme point of the polytope obtained from $R(A)$ by deleting the inequalities that are not satisfied with equality by $\bar{x}$. By Theorem 6.12, every extreme point of this polytope has $0, 1$ components. $\Box$

Corollary 6.14 *If A is a balanced $0, \pm 1$ matrix, the generalized set covering polytope with constraint matrix A is an integral polytope.*

Berge showed that every balanced $0, 1$ matrix is perfect.

Exercise 6.15 *Prove that every balanced 0,1 matrix is perfect, using Theorem 6.13.*

The next theorem gives three equivalent characterizations of balanced $0, 1$ matrices.

Theorem 6.16 (Berge [7], Fulkerson, Hoffman, Oppenheim [92]) *Let A be a $0, 1$ matrix. Then the following statements are equivalent:*
(i) A is balanced.

(ii) Every submatrix of A is perfect.
(iii) Every submatrix of A is ideal.
(iv) For each submatrix B of A, the set partitioning polytope $\{x: Bx = \mathbf{1},\ x \geq 0\}$ is integral.

Proof: This result is a corollary of Theorem 6.13. $\square$

6.2.2 Total Dual Integrality

Theorem 6.17 (Fulkerson, Hoffman, Oppenheim [92]) *Let* $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ *be a balanced* $0,1$ *matrix. Then the linear system*

$$\left\{ \begin{array}{rcl} A_1 x & \geq & \mathbf{1} \\ A_2 x & \leq & \mathbf{1} \\ A_3 x & = & \mathbf{1} \\ x & \geq & 0 \end{array} \right.$$

is TDI.

Exercise 6.18 *Use Theorem 6.17 to prove the following result of Berge and Las Vergnas [13], which generalizes König's theorem (Theorem 1.1). If M is a balanced clutter matrix, then the maximum number of disjoint edges in $\mathcal{C}(M)$ equals the minimum cardinality of a transversal.*

Theorem 6.17 and the Edmonds-Giles theorem imply Theorem 6.16. In this section, we prove the following more general result.

Theorem 6.19 (Conforti, Cornuéjols [42]) *Let* $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ *be a balanced* $0, \pm 1$ *matrix. Then the linear system*

$$\left\{ \begin{array}{rcl} A_1 x & \geq & \mathbf{1} - n(A_1) \\ A_2 x & \leq & \mathbf{1} - n(A_2) \\ A_3 x & = & \mathbf{1} - n(A_3) \\ 0 & \leq & x \leq \mathbf{1} \end{array} \right.$$

is TDI.

The following transformation of a $0, \pm 1$ matrix A into a $0,1$ matrix B is often seen in the literature: to every column a_j of A, $j = 1, \ldots, p$, associate two columns of B, say b_j^P and b_j^N, where $b_{ij}^P = 1$ if $a_{ij} = 1$, 0 otherwise, and $b_{ij}^N = 1$ if $a_{ij} = -1$, 0 otherwise. Let D be the $0,1$ matrix with p rows and $2p$ columns d_j^P and d_j^N such that $d_{jj}^P = d_{jj}^N = 1$ and $d_{ij}^P = d_{ij}^N = 0$ for $i \neq j$.

Given a $0, \pm 1$ matrix $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ and the associated $0,1$ matrix $B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$, define the following linear systems:

$$\left\{ \begin{array}{rcl} A_1 x & \geq & \mathbf{1} - n(A_1) \\ A_2 x & \leq & \mathbf{1} - n(A_2) \\ A_3 x & = & \mathbf{1} - n(A_3) \\ 0 & \leq & x \ \leq \ \mathbf{1}, \end{array} \right. \tag{6.1}$$

$$\text{and} \left\{ \begin{array}{rcl} B_1 y & \geq & \mathbf{1} \\ B_2 y & \leq & \mathbf{1} \\ B_3 y & = & \mathbf{1} \\ D y & = & \mathbf{1} \\ y & \geq & 0. \end{array} \right. \tag{6.2}$$

The vector $x \in R^p$ satisfies (6.1) if and only if the vector $(y^P, y^N) = (x, \mathbf{1} - x)$ satisfies (6.2). Hence the polytope defined by (6.1) is integral if and only if the polytope defined by (6.2) is integral. We show that, if A is a balanced $0, \pm 1$ matrix, then both (6.1) and (6.2) are TDI.

Lemma 6.20 *If* $A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$ *is a balanced* $0, \pm 1$ *matrix, the corresponding system (6.2) is TDI.*

Proof: The proof is by induction on the number m of rows of B. Let $c = (c^P, c^N) \in Z^{2p}$ denote an integral vector and R_1, R_2, R_3 the index sets of the rows of B_1, B_2, B_3 respectively. The dual of min $\{cy :$

y satisfies (6.2)$\}$ is the linear program

$$\begin{array}{llll} \max & \sum_{i=1}^m u_i & +\sum_{j=1}^p v_j & \\ & uB & +vD & \leq c \\ & u_i \geq 0, & i \in R_1 & \\ & u_i \leq 0, & i \in R_2. & \end{array} \tag{6.3}$$

Since v_j only appears in two of the constraints $uB + vD \leq c$ and no constraint contains v_j and v_k, it follows that any optimal solution to (6.3) satisfies

$$v_j = \min\,(c_j^P - \sum_{i=1}^m b_{ij}^P u_i,\ c_j^N - \sum_{i=1}^m b_{ij}^N u_i). \tag{6.4}$$

Let $(\bar{u}, \bar{v})$ be an optimal solution of (6.3). If $\bar{u}$ is integral, then so is $\bar{v}$ by (6.4) and we are done. So assume that $\bar{u}_\ell$ is fractional. Let b^ℓ be the corresponding row of B and let B_ℓ be the matrix obtained from B by removing row b^ℓ. By induction on the number of rows of B, the system (6.2) associated with B_ℓ is TDI. Hence the system

$$\begin{array}{llll} \max & \sum_{i \neq \ell} u_i & +\sum_{j=1}^p v_j & \\ & u_\ell B_\ell & +vD & \leq c - \lfloor \bar{u}_\ell \rfloor b^\ell \\ & u_i \geq 0, & i \in R_1 \setminus \{\ell\} & \\ & u_i \leq 0, & i \in R_2 \setminus \{\ell\} & \end{array} \tag{6.5}$$

has an integral optimal solution $(\tilde{u}, \tilde{v})$. Since $(\bar{u}_1, \ldots, \bar{u}_{\ell-1}, \bar{u}_{\ell+1}, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_p)$ is a feasible solution to (6.5) and Theorem 6.13 shows that $\sum_{i=1}^m \bar{u}_i + \sum_{j=1}^p \bar{v}_j$ is an integer number,

$$\sum_{i \neq \ell} \tilde{u}_i + \sum_{j=1}^p \tilde{v}_j \geq \lceil \sum_{i \neq \ell} \bar{u}_i + \sum_{j=1}^p \bar{v}_j \rceil = \sum_{i=1}^m \bar{u}_i + \sum_{j=1}^p \bar{v}_j - \lfloor \bar{u}_\ell \rfloor.$$

Therefore the vector $(u^*, v^*) = (\tilde{u}_1, \ldots, \tilde{u}_{\ell-1}, \lfloor \bar{u}_\ell \rfloor, \tilde{u}_{\ell+1}, \ldots, \tilde{u}_m, \tilde{v}_1, \ldots, \tilde{v}_p)$ is integral, is feasible to (6.3) and has an objective function value not smaller than $(\bar{u}, \bar{v})$, proving that the system (6.2) is TDI. $\square$

Proof of Theorem 6.19: Let R_1, R_2, R_3 be the index sets of the rows of A_1, A_2, A_3. By Lemma 6.20, the linear system (6.2) associated with (6.1) is TDI. Let $d \in R^p$ be any integral vector. The dual of $\min\{dx : x \text{ satisfies } (6.1)\}$ is the linear program

$$\begin{array}{lll} \max & w(\mathbf{1} - n(A)) & -t\mathbf{1} \\ & wA & -t \ \ \leq d \\ & w_i \geq 0, & i \in R_1 \\ & w_i \leq 0, & i \in R_2 \\ & t \geq 0. & \end{array} \tag{6.6}$$

For every feasible solution $(\bar{u}, \bar{v})$ of (6.3) with $c = (c^P, c^N) = (d, 0)$, we construct a feasible solution $(\bar{w}, \bar{t})$ of (6.6) with the same objective function value as follows:

$$\begin{array}{rcl} \bar{w} & = & \bar{u} \\ \bar{t}_j & = & \left\{ \begin{array}{lll} 0 & if & \bar{v}_j = -\sum_i b^N_{ij}\bar{u}_i \\ \sum_i b^P_{ij}\bar{u}_i - \sum_i b^N_{ij}\bar{u}_i - d_j & if & \bar{v}_j = d_j - \sum_i b^P_{ij}\bar{u}_i. \end{array} \right. \end{array} \tag{6.7}$$

When the vector $(\bar{u}, \bar{v})$ is integral, the above transformation yields an integral vector $(\bar{w}, \bar{t})$. Therefore (6.6) has an integral optimal solution and the linear system (6.1) is TDI. $\square$

Exercise 6.21 *Give an example showing that Theorem 6.19 does not hold when the upper bound $x \leq \mathbf{1}$ is dropped from the linear system.*

6.2.3 A Bicoloring Theorem

Berge [6] introduced the following notion. A $0,1$ matrix is *bicolorable* if its columns can be partitioned into blue and red columns in such a way that every row with two or more $1's$ contains a 1 in a blue column and a 1 in a red column.

Theorem 6.22 (Berge [6]) *A $0,1$ matrix A is balanced if and only if every submatrix of A is bicolorable.*

We prove this result in a more general form (Theorem 6.26). A $0, \pm 1$ matrix A is *bicolorable* if its columns can be partitioned into blue columns and red columns in such a way that every row with two or more nonzero entries either contains two entries of opposite sign in columns of the same color, or contains two entries of the same sign in columns of different colors. For a $0, 1$ matrix, this definition coincides with Berge's definition. Clearly, if a $0, \pm 1$ matrix has an equitable bicoloring as defined by Ghouila-Houri, then it is bicolorable. Recall that equitable bicolorings can be used to characterize totally unimodular matrices (Theorem 6.3).

Proposition 6.23 (Heller, Tompkins [120]) *Let A be a $0, \pm 1$ matrix with at most two nonzero entries per row. A is totally unimodular if and only if A is bicolorable.*

Exercise 6.24 *Prove Proposition 6.23.*

Exercise 6.25 *Show that a $0, \pm 1$ matrix with at most two nonzero entries per row is balanced if and only if it is totally unimodular.*

So Proposition 6.23 shows that a $0, \pm 1$ matrix with at most two nonzero entries per row is balanced if and only if it is bicolorable. The following theorem extends Theorem 6.22 to $0, \pm 1$ matrices and Proposition 6.23 to matrices with more than two nonzero entries per row.

Theorem 6.26 (Conforti, Cornuéjols [42]) *A $0, \pm 1$ matrix A is balanced if and only if every submatrix of A is bicolorable.*

Proof: Assume first that A is balanced and let B be any submatrix of A. Remove from B any row with fewer than two nonzero entries. Since B is balanced, so is the matrix $(B, -B)$. It follows from Corollary 6.14 that

$$\begin{array}{rcl} Bx & \geq & \mathbf{1} - n(B) \\ -Bx & \geq & \mathbf{1} - n(-B) \\ 0 \leq x & \leq & \mathbf{1} \end{array} \tag{6.8}$$

is an integral polytope. Since it is nonempty (the vector $(1/2, \dots, 1/2)$ is a solution), it contains a 0,1 point $\bar{x}$. Color a column j of B red if $\bar{x}_j = 1$ and blue otherwise. By (6.8), this is a valid bicoloring of B.

Conversely, assume that A is not balanced. Then A contains an unbalanced hole matrix B and by Proposition 6.23, B is not bicolorable. $\square$

Cameron and Edmonds [25] observed that the following simple algorithm finds a valid bicoloring of a balanced matrix.

Algorithm

Input: A $0, \pm 1$ matrix A.

Output: A bicoloring of A or a proof that the matrix A is not balanced.

Stop if all columns are colored or if some row is incorrectly colored. Otherwise, color a new column red or blue as follows.

If no row of A forces the color of a column, arbitrarily color one of the uncolored columns.

If some row of A forces the color of a column, color this column accordingly.

When the algorithm fails to find a bicoloring, the sequence of forcings that resulted in an incorrectly colored row identifies a unbalanced hole submatrix of A. The algorithm may find a correct bicoloring even when A is not balanced. For example, if $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$, the algorithm may color the first two columns blue and the last two red, which is a correct bicoloring of A. For this reason, the algorithm cannot be used as a recognition of balancedness.

6.3 Perfect and Ideal $0, \pm 1$ Matrices

A $0, \pm 1$ matrix A is *perfect* if its fractional generalized set packing polytope $P(A) = \{x : \; Ax \leq \mathbf{1} - n(A), \; 0 \leq x \leq \mathbf{1}\}$ is integral. Similarly, a $0, \pm 1$ matrix A is *ideal* if its fractional generalized set covering polytope $Q(A) = \{x : \; Ax \geq \mathbf{1} - n(A), \; 0 \leq x \leq \mathbf{1}\}$ is integral.

6.3.1 Relation to Perfect and Ideal 0,1 Matrices

Hooker [125] was the first to relate idealness of a $0, \pm 1$ matrix to that of a family of 0,1 matrices. A similar result for perfection was obtained in [44]. These results were strengthened by Guenin [109] and by Boros, Čepek [18] for perfection, and by Nobili, Sassano [148] for idealness. The key tool for these results is the following:

Given a $0, \pm 1$ matrix A, let P and R be $0, 1$ matrices of the same dimension as A, with entries $p_{ij} = 1$ if and only if $a_{ij} = 1$, and $r_{ij} = 1$ if and only if $a_{ij} = -1$. The matrix

$$D_A = \begin{pmatrix} P & R \\ I & I \end{pmatrix},$$

is the $0, 1$ *extension* of A. Note that the transformation $x^+ = x$ and $x^- = \mathbf{1} - x$ maps every vector x in $P(A)$ into a vector in $\{(x^+, x^-) \geq 0 : \; Px^+ + Rx^- \leq \mathbf{1}, \; x^+ + x^- = \mathbf{1}\}$ and every vector x in $Q(A)$ into a vector in $\{(x^+, x^-) \geq 0 : \; Px^+ + Rx^- \geq \mathbf{1}, \; x^+ + x^- = \mathbf{1}\}$. So $P(A)$ and $Q(A)$ are respectively the faces of $P(D_A)$ and $Q(D_A)$, obtained by setting the inequalites $x^+ + x^- \leq \mathbf{1}$ and $x^+ + x^- \geq \mathbf{1}$ at equality.

Given a $0, \pm 1$ matrix A, let a^1 and a^2 be two rows of A, such that there is one index k such that $a^1_k a^2_k = -1$ and, for all $j \neq k$, $a^1_j a^2_j = 0$. A *disjoint implication* of A is the $0, \pm 1$ vector $a^1 + a^2$. The matrix A^+ obtained by recursively adding all disjoint implications and removing all dominated rows (in the packing case) or dominating rows (in the covering case) is called the *disjoint completion* of A.

Theorem 6.27 (Nobili, Sassano [148]) *Let A be a $0, \pm 1$ matrix. Then A is ideal if and only if the 0,1 matrix D_{A^+} is ideal.*

Furthermore A is ideal if and only if $\min\{cx : \; x \in Q(A)\}$ has an integer optimum for every vector $c \in \{0, \pm 1, \pm\infty\}^n$.

Theorem 6.28 (Guenin [109]) *If A is a $0, \pm 1$ matrix, where $P(A)$ is not contained in any of the hyperplanes $\{x : x_j = 0\}$ or $\{x : x_j = 1\}$, then A is perfect if and only if the $0, 1$ matrix D_{A^+} is perfect.*

Theorem 6.29 (Guenin [109]) *A $0, \pm 1$ matrix A is perfect if and only if $\max\{cx : \; x \in P(A)\}$ admits an integral optimal solution for every $c \in \{0, \pm 1\}^n$. Moreover, if A is perfect, the linear system $Ax \leq \mathbf{1} - n(A)$, $0 \leq x \leq \mathbf{1}$ is TDI.*

This is the natural extension of the Lovász's theorem for perfect $0, 1$ matrices.

6.3.2 Propositional Logic

In propositional logic, a *boolean variable* x_j can take the value *true* or *false*. For a set of boolean variables $x_1, \ldots, x_j, \ldots, x_n$, a *truth assignment* is an assignment of "true" or "false" to each boolean variable. A *literal* is a boolean variable x_j or its negation $\neg x_j$. A *clause* is a disjunction of literals and is *satisfied* by a given truth assignment if at least one of its literals is true.

A survey of the connections between propositional logic and integer programming can be found in Hooker [124], Truemper [196] or Chandru and Hooker [29].

A truth assignment satisfies the set S of clauses

$$\bigvee_{j \in P_i} x_j \vee (\bigvee_{j \in N_i} \neg x_j) \quad \text{for all} \;\; i \in S$$

if and only if the corresponding $0,1$ vector satisfies the system of inequalities

$$\sum_{j \in P_i} x_j - \sum_{j \in N_i} x_j \geq 1 - |N_i| \quad \text{for all} \;\; i \in S.$$

The above system of inequalities is of the form

$$Ax \geq \mathbf{1} - n(A). \tag{6.9}$$

In [43], we consider three classical problems in logic. Given a set S of clauses, the *satisfiability problem (SAT)* consists in finding a truth assignment that satisfies all the clauses in S or show that none exists. Equivalently, SAT consists in finding a $0,1$ solution x to (6.9) or show that none exists.

Given a set S of clauses and a weight vector w whose components are indexed by the clauses in S, the *weighted maximum satisfiability problem (MAXSAT)* consists in finding a truth assignment that maximizes the total weight of the satisfied clauses. MAXSAT can be formulated as the integer program

$$\begin{array}{ll} \min & \sum_{i=1}^m w_i s_i \\ & Ax + s \geq \mathbf{1} - n(A) \\ & x \in \{0,1\}^n, s \in \{0,1\}^m. \end{array}$$

Given a set S of clauses (the premises) and a clause C (the conclusion), *logical inference* in propositional logic consists of deciding whether every truth assignment that satisfies all the clauses in S also satisfies the conclusion C.

To the clause C, using transformation (6.9), we associate an inequality

$$cx \geq 1 - n(c),$$

where c is a $0, +1, -1$ vector. Therefore C cannot be deduced from S if and only if the integer program

$$\min \{cx : Ax \geq \mathbf{1} - n(A),\ x \in \{0,1\}^n\} \tag{6.10}$$

has a solution with value $-n(c)$.

The above three problems are NP-hard in general but SAT and logical inference can be solved efficiently for Horn clauses, clauses with at most two literals and several related classes [28],[193]. MAXSAT remains NP-hard for Horn clauses with at most two literals [98]. A set S of clauses is *ideal* (*balanced* respectively) if the corresponding $0, \pm 1$ matrix A defined in (6.9) is ideal (balanced respectively). If S is ideal, it follows from the definition that the satisfiability and logical inference problems can be solved by linear programming. The following theorem is an immediate consequence of Corollary 6.14.

Theorem 6.30 *Let S be a balanced set of clauses. Then the satisfiability, MAXSAT and logical inference problems can be solved in polynomial time by linear programming.*

This has consequences for probabilistic logic as defined by Nilsson [147]. Being able to solve MAXSAT in polynomial time provides a polynomial time separation algorithm for probabilistic logic via the ellipsoid method, as observed by Georgakopoulos, Kavvadias and Papadimitriou [98]. Hence probabilistic logic is solvable in polynomial time for balanced sets of clauses.

Remark 6.31 *Let S be an ideal set of clauses. If every clause of S contains more than one literal then, for every boolean variable x_j, there exist at least two truth assignments satisfying S, one in which x_j is true and one in which x_j is false.*

Proof: Since the point $x_j = 1/2,\ j = 1, \ldots, n$ belongs to the polytope $Q(A) = \{x : Ax \geq \mathbf{1} - n(A),\ 0 \leq x \leq \mathbf{1}\}$ and $Q(A)$ is an integral polytope, then the above point can be expressed as a convex combination of $0, 1$ vectors in $Q(A)$. Clearly, for every index j, there exists a $0, 1$ vector in the convex combination with $x_j = 0$ and another with $x_j = 1$. $\square$

Let S be an ideal set of clauses. A consequence of Remark 6.31 is that the satisfiability problem can be solved more efficiently than by general linear programming.

Theorem 6.32 (Conforti, Cornuéjols [43]) *Let S be an ideal set of clauses. Then S is satisfiable if and only if a recursive application of the following procedure stops with an empty set of clauses.*

Recursive Step

If $S = \emptyset$, then S is satisfiable.

If S contains a clause C with a single literal (unit clause)*, set the corresponding boolean variable x_j so that C is satisfied. Eliminate from S all clauses that become satisfied and remove x_j from all the other clauses. If a clause becomes empty, then S is not satisfiable* (unit resolution)*.*

If every clause in S contains at least two literals, choose any boolean variable x_j appearing in a clause of S and add to S an arbitrary clause x_j or $\neg x_j$.

Exercise 6.33 *Modify the above algorithm in order to solve the logical inference problem when S is an ideal set of clauses.*

For balanced (or ideal) sets of clauses, it is an open problem to solve the MAXSAT problem in polynomial time by a direct method, without appealing to polynomial time algorithms for general linear programming.

6.3.3 Bigraphs and Perfect $0, \pm 1$ Matrices

This section follows [48]. An inequality $ax \leq 1 - n(a)$ of a generalized set packing problem can be written as $\sum_{i \in P} x_i + \sum_{i \in N}(1 - x_i) \leq 1$, where P is the set of indices i where $a_i = 1$ and N is the set of indices where

$a_i = -1$. Therefore the set of $0,1$ solutions of the above inequality is exactly the set of $0,1$ solutions of the following system of 2-variable boolean inequalities:

$$x_i + x_j \leq 1, \quad \text{for all } i, j \in P$$

$$x_i + x_j \geq 1, \quad \text{for all } i, j \in N$$

$$x_i \leq x_j, \quad \text{for all } i \in P, \ j \in N$$

Therefore, if A is a $0, \pm 1$ matrix, the set of $0,1$ vectors in the generalized set packing polytope $P(A) \equiv \{x \geq 0 : \ Ax \leq \mathbf{1} - n(A)\}$ is also the set of $0,1$ solutions of a system of 2-variable boolean inequalities (S2BI).

We can model these inequalities with a *bigraph* $B = (V; P, N, M)$ where the set of nodes V represents the variables, edges in P represent the inequalities of the first type and have two $+$ signs at their ends, edges in N represent the inequalities of the second type and have two $-$ signs at their ends and edges in M represent the inequalities of the third type and have a $+$ in correspondence to the endnode i and a $-$ sign for endnode j. So the characteristic vector of $S \subseteq V$ is a solution of a S2BI if and only if S is a stable set of the graph $B = (V; P, \emptyset, \emptyset)$, a node cover of the graph $B = (V; \emptyset, N, \emptyset)$ and satisfies the precedences of $B = (V; \emptyset, \emptyset, M)$.

Note that a S2BI may imply additional 2-variable boolean inequalities. For example, the inequalities $x_i + x_j \leq 1$ and $x_k \leq x_j$ imply the inequality $x_i + x_k \leq 1$. In terms of the bigraph, these inequalities corresponds to additional edges. We define the *transitive closure* of a S2BI as the set of boolean constraints satisfied by the $0,1$ solutions of the original system. A S2BI is *closed* if it coincides with its transitive closure.

Also, a S2BI may fix the value of a variable to 0 or 1, or to the value of another variable. A S2BI is *reduced* if no variable is fixed to 0 or to 1, no two variables are identically equal and no pair of variables sum to 1. When a S2BI is not reduced, some variables may be eliminated.

Johnson and Padberg [128] show how to compute the closure of a S2BI and to test whether a S2BI is reducible. (This is also known in the context of boolean optimization).

A *biclique* K in a bigraph is a subset of mutually adjacent nodes such that no two edges of K meet the same endnode with distinct signs. So the nodes of K are partitioned in K^+ and K^-, according to the signs of the edges of K. The inequality:

$$\sum_{i \in K^+} x_i + \sum_{i \in K^-} (1 - x_i) \leq 1$$

is clearly valid for the $0, 1$ vectors in a S2BI. Johnson and Padberg show that it is facet-inducing for the convex hull of 0, 1 solutions of S2BI if and only if K is a maximal biclique.

Johnson and Padberg define a bigraph with set of maximal bicliques $\mathcal{K}$ to be *perfect* if both the linear program

$$\max \{bx : \ 0 \leq x \leq \mathbf{1}, \ \ \sum_{i \in K^+} x_i + \sum_{i \in K^-} (1 - x_i) \leq 1 \ \text{ for all } K \in \mathcal{K}\}$$

and its dual admit optimal solutions that are integral whenever b is a $0, \pm 1$-valued vector.

Bidirecting the edges of an undirected graph G means assigning $+$ or $-$ signs to the endnodes of each edge. G is *biperfect* if every bigraph that is closed and reduced and is obtained by bidirecting the edges of G, is perfect. Sewell [179] proved the following results, conjectured by Johnson and Padberg [128] ten years earlier.

Theorem 6.34 (Sewell [179]) *A graph is biperfect if and only if it is perfect.*

Theorem 6.35 (Sewell [179]) *Let G be a graph. If there exists a bidirection of the edges of G that gives a closed, reduced bigraph that is perfect, then G is perfect.*

Ikebe and Tamura [127] and Li [134] give another, independent, proof of these theorems. Since bidirecting the edges of a graph with $+, +$ at their endnodes gives a closed and reduced bigraph, every biperfect graph is perfect. Therefore Theorem 6.35 implies Theorem 6.34.

Chapter 7

Signing 0,1 Matrices to be TU or Balanced

A $0,1$ matrix is *regular* if its nonzero entries can be signed $+1$ or -1 so that the resulting matrix is totally unimodular. A result of Camion [27] shows that this signing is unique up to changing signs in rows or columns. So the recognition of totally unimodular matrices reduces to that of regular matrices. Similarly, a $0,1$ matrix is *balanceable* if its nonzero entries can be signed $+1$ or -1 so that the resulting $0,\pm 1$ matrix is balanced. In this lecture, we present Tutte's characterization [202] of regular matrices and Truemper's characterization [192] of balanceable matrices.

7.1 Camion's Signing Algorithm

A $0,1$ matrix is *regular* (or *signable to be totally unimodular*) if its nonzero entries can be signed $+1$ or -1 in such a way that the resulting $0,\pm 1$ matrix is totally unimodular.

Camion [26] observed that the signing of a regular matrix into a totally unimodular matrix is unique up to multiplying rows or columns by -1. Furthermore there is a simple algorithm to obtain the signing. Therefore, although our primary interest is in total unimodularity, it is convenient to work with regular matrices, i.e. with the pattern of zero/nonzero entries, without being concerned with the signs of the nonzero entries. We present Camion's result next.

Let A be a $0, \pm 1$ matrix and let A' be obtained from A by multiplying a set S of rows and columns by -1. A is totally unimodular if and only if A' is. Note that, in the bipartite representation of A, this corresponds to switching signs on all edges of the cut $(S, \bar{S})$. Now let R be a 0,1 matrix and $B(R)$ is its bipartite representation. Since every edge of a maximal forest F of $B(R)$ is contained in a cut that does not contain any other edge of F, it follows that if R is regular, there exists a totally unimodular signing of R in which the edges of F have any specified (arbitrary) signing.

This implies that, if a 0,1 matrix A is regular, one can produce a totally unimodular signing of A as follows.

CAMION'S SIGNING ALGORITHM

Input: *A regular 0,1 matrix and its bipartite representation G, a maximal forest F and an arbitrary signing of the edges of F.*

Output: *The unique totally unimodular signing of G such that the edges of F are signed as specified in the input.*

Index the edges of G $e_1, \dots, e_n$, so that the edges of F come first, and every edge e_j, $j \geq |F|+1$, together with edges having smaller indices, closes a chordless cycle H_j of G. For $j = |F|+1, \dots, n$, sign e_j so that H_j is totally unimodular.

The fact that there exists an indexing of the edges of G as required in the signing algorithm follows from the following observation. For $j \geq |F|+1$, we can select e_j so that the path connecting the endnodes of e_j in the subgraph $(V(G), \{e_1, \dots, e_{j-1}\})$ is shortest possible. The chordless cycle H_j identified this way is also a chordless cycle in G. This forces the signing of e_j, since all the other edges of H_j are signed already (recall Exercise 6.8). So, once the (arbitrary) signing of F has been chosen, the signing of G is unique.

Exercise 7.1 *Is the following matrix regular?*

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Assume that we have an algorithm to check regularity. Then, given a signed bipartite graph G, we can check whether G is totally unimodular as follows. Let G' be an unsigned copy of G. Test whether G' is regular. If it is not, then G is not totally unimodular. Otherwise, let F be a maximal forest of G'. Run the signing algorithm on G' with the edges of F signed as they are in G. Then G is totally unimodular if and only if the signing of G' coincides with the signing of G.

7.2 Pivoting

Consider the classical operation of *pivoting* on a nonzero entry a of a matrix A in compact form, i.e. the rows of A index the basic variables and the columns the nonbasic. Pivoting on $a \neq 0$ consists of replacing the matrix

$$A = \begin{pmatrix} a & y \\ x & D \end{pmatrix} \quad \text{by} \quad \tilde{A} = \begin{pmatrix} 1/a & y/a \\ -x/a & D - xy/a \end{pmatrix}$$

where x is a column vector, y is a row vector and D is a matrix. Note that, by pivoting on the entry $1/a$ in the matrix $\tilde{A}$, the initial matrix A is restored.

Exercise 7.2 *Let A be a $0, \pm 1$ matrix. Show that all the matrices obtained from A by pivoting are $0, \pm 1$ matrices if and only if A contains no 2×2 unbalanced submatrix.*

Remark 7.3 *In linear programming textbooks, pivoting is usually defined with respect to the full tableau $R = (I, A)$ rather than the compact form A. The identity matrix I defines the basic columns. In this context, pivoting on $a_{ij} \neq 0$ means performing elementary row operations on R that transform each row r^k into $r^k + \lambda_k r^i$ so that the jth column of A becomes the unit vector e^i. Permuting this column with the ith column of I, which has become nonbasic, the matrix (I, A) becomes $(I, \tilde{A})$. It is easy to check that this matrix $\tilde{A}$ corresponds to the above definition of pivoting.*

Exercise 7.4 *Show that, if A is a square $0, \pm 1$ matrix, then $det(A) = \pm det(D - xy/a)$.*

Proposition 7.5 *$\tilde{A}$ is totally unimodular if and only if A is.*

Proof: $A = \begin{pmatrix} a & y \\ x & D \end{pmatrix}$ is totally unimodular if and only if $B = \begin{pmatrix} 1 & a & y \\ \mathbf{0} & x & D \end{pmatrix}$ is. Similarly, $\tilde{A} = \begin{pmatrix} 1/a & y/a \\ -x/a & D - xy/a \end{pmatrix}$ is totally unimodular if and only if $\tilde{B} = \begin{pmatrix} 1/a & 1 & y/a \\ -x/a & \mathbf{0} & D - xy/a \end{pmatrix}$ is. To prove the proposition, it suffices to show that, if B is totally unimodular, then so is $\tilde{B}$. Consider a square submatrix $\tilde{M}$ of $\tilde{B}$ and the corresponding matrix M of B (i.e. with same set of rows and columns). If $\tilde{M}$ contains row 1, then $det(\tilde{M}) = det(M)$ since $\tilde{B}$ is obtained from B by elementary row operations involving row 1. If $\tilde{M}$ does not contain row 1 but contains column 2, then $det(\tilde{M}) = 0$. Finally, if $\tilde{M}$ contains neither row 1 nor column 2, then let $\tilde{N}$ be obtained from $\tilde{M}$ by adding this row and column. Since $\tilde{N}$ contains a unit column, $det(\tilde{N}) = det(\tilde{M})$. Now, since $\tilde{N}$ contains row 1, $det(\tilde{N}) = det(N)$, where N is the corresponding matrix of B. It follows that $det(\tilde{M}) = det(N)$. So, in all three cases, $det(\tilde{M}) = 0, \pm 1$ if and only if B is totally unimodular. $\square$

Corollary 7.6 *If A is a $k \times k$ minimally non totally unimodular $0, \pm 1$ matrix where $k \geq 3$, then the submatrix $D - xy/a$ of $\tilde{A}$ is also a minimally non totally unimodular $0, \pm 1$ matrix.*

Proof: If A is an unbalanced hole, then $D - xy/a$ is also an unbalanced hole and the corollary holds. Assume now that A is balanced. By Exercise 7.2, $\tilde{A}$ is a $0, \pm 1$ matrix and by Exercise 7.4, $D - xy/a$ is not totally unimodular. However, by Proposition 7.5, any proper submatrix of $D - xy/a$ is totally unimodular, since it is obtained from a proper submatrix of A by pivoting. $\square$

The pivoting operation can be performed over any field. Here, we will consider pivoting over GF(2). In this case, the pivoting operation consists of replacing the matrix

$$A = \begin{pmatrix} 1 & y \\ x & D \end{pmatrix} \quad \text{by} \quad \tilde{A} = \begin{pmatrix} 1 & y \\ x & D + xy \end{pmatrix}.$$

Exercise 7.7 *Let $B(A)$ be the bipartite representation of A and $e = uv$ be the edge corresponding to the pivot element. Show that the bipartite representation $B(\tilde{A})$ of $\tilde{A}$ is obtained from $B(A)$ by complementing the edges between $N(u) - \{v\}$ and $N(v) - \{u\}$.*

Proposition 7.5 implies the following result.

Corollary 7.8 *If A is a regular $0, 1$ matrix, then any matrix $\tilde{A}$ obtained from A by a sequence of pivots over GF(2) is regular.*

Exercise 7.9 *Prove Corollary 7.8.*

7.3 Tutte's Theorem

Denote the matrix of Exercise 7.1 by $R(F_7)$.

Theorem 7.10 (Tutte [202]) *A 0,1 matrix A is regular if and only if it cannot be transformed to $R(F_7)$ by applying the following operations:*

- *pivoting over GF(2),*
- *deleting rows or columns,*
- *permuting rows or columns,*
- *taking the transpose matrix.*

Tutte's original proof [202] is complicated. We present here a short proof due to Gerards [99]. For a $0, 1$ matrix A, let $B(A)$ be the bipartite graph whose nodes correspond to the rows and columns of A and where ij is an edge of $B(A)$ if and only if the entry a_{ij} of A equals 1. We say that the bipartite graph $B(A)$ is *regular* if A is regular.

We will use the following observation, which follows from Exercise 7.7.

Proposition 7.11 *Let A' be obtained from a $0, 1$ matrix A by pivoting. If $B(A)$ is a connected graph, then so is $B(A')$.*

Proof of Theorem 7.10: (Gerards [99]) The necessity is easy since $R(F_7)$ is not regular and the operations in the statement of the theorem all preserve regularity. So it remains to prove sufficiency.

Suppose A is a 0,1 matrix that is not regular. We may assume that each proper submatrix of A is regular. It then follows that $B(A)$ is a connected graph. Since $B(A)$ is not regular, it is not a path or a circuit. Let T be a spanning tree of $B(A)$ which is not a path and let u, v be two leaves of T in the same side of the bipartition, say both correspond to columns of A. Let x and y denote these two columns. Clearly, $B(A) \setminus \{u, v\}$ is connected. Let N be the corresponding matrix. By assumption, (x, N) and (y, N) are regular. The two corresponding signed matrices being uniquely defined, up to switching signs on rows or columns, we can assume that N is signed in the same way in both. Let the signed matrices be (x', N') and (y', N') respectively. Then

- $B(N)$ is connected.
- Both matrices $(x'N')$ and $(y'N')$ are totally unimodular.

Both of these properties are preserved by pivoting on nonzero elements of N'. Now pivot on nonzero entries of N' in such a way that the smallest square submatrix of A' with determinant distinct from $0, +1$ or -1 is as small as possible. Such a submatrix M must be a 2x2 matrix. (If not, pivot on a nonzero entry of M which belongs to N'. Then, by Exercise 7.4, we get a contradiction to the choice of M.) Since $(x'N')$ and $(y'N')$ are totally unimodular, M must be a submatrix of $(x'y')$. Furthermore, $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ after possibly multiplying rows and columns of A' by -1. In the remainder, we assume w.l.o.g. that A' denotes the matrix after pivoting, i.e. it contains the above submatrix M.

Denote by s and t the nodes of $B(N)$ which correspond to the rows of M, and consider a chordless path P from s to t in $B(N)$. The path P cannot have length 2 since this would imply a 2x2 submatrix with determinant ± 2 in $(x'N')$ or $(y'N')$, contradicting the fact that these matrices are totally unimodular. Now assume the path P has length 6 or more, say $P = s, e_1, v_1, e_2, v_2, e_3, \ldots, v_{2k-1}, e_{2k}, t$, where $k \geq 3$ and

$v_1, v_3, \ldots, v_{2k-1}$ are column nodes, $v_2, \ldots, v_{2k-2}$ are row nodes. Then pivot on the nonzero entry of A' corresponding to e_3. The only entry that is modified in the submatrix of N' induced by rows $v_2, \ldots, v_{2k-2}$ and columns $v_1, v_3, \ldots, v_{2k-1}$ is the entry in row v_4 and column v_1, which was 0 before pivoting and becomes nonzero. Now delete row v_2 and column v_3. In the new graph, there is still a chordless path from s to t but its length has decreased by 2. So we can assume w.l.o.g. that P has length 4. Its nodes together with nodes u and v induce the following submatrix of A'

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ \alpha & \beta & 1 & 1 \end{pmatrix}$$

after permuting rows and columns and multiplying columns 3 and 4, and rows 2 and 3 by -1 if necessary. It is still the case that removing either of the first two columns yields a totally unimodular matrix. This implies $\alpha = 1$ and $\beta = 0$. Hence A can be transformed to $R(F_7)$. $\Box$

Gerards [101] used Tutte's theorem to characterize the undirected graphs that admit an edge orientation such that, going around any circuit in the graph, the number of forward edges minus the number of backward edges is equal to $0, \pm 1$.

Geelen [97] obtained a generalization of Tutte's theorem. He characterized the symmetric $0, \pm 1$ matrices in which all principal submatrices have $0, \pm 1$ determinants.

7.4 Truemper's Theorem

A 0,1 matrix A is *balanceable* if its nonzero entries can be signed +1 or -1 in such a way that the resulting $0, \pm 1$ matrix is balanced. Camion's signing algorithm (Section 7.1) can be used to find such a signing if one exists:

Theorem 7.12 *If the input matrix is a balanceable 0,1 matrix, Camion's signing algorithm produces a balanced* $0, \pm 1$ *matrix. Furthermore this signing is unique up to switching signs on rows and columns.*

Exercise 7.13 *Prove Theorem 7.12, using the arguments of Section 7.1.*

Corollary 7.14 *Every balanced signing of a regular matrix is totally unimodular.*

As a consequence of Theorem 7.12, one can check balancedness of a $0, \pm 1$ matrix in polytime if and only if one can check balanceability of a 0,1 matrix in polytime.

In a bipartite graph, a *wheel* (H, v) consists of a hole H and a node v having at least three neighbors in H. The wheel (H, v) is *odd* if v has an odd number of neighbors in H. A *3-path configuration* is an induced subgraph consisting of three internally node-disjoint paths connecting two nonadjacent nodes u and v and containing no edge other than those of the paths. If u and v are in opposite sides of the bipartition, i.e. the three paths have an odd number of edges, the 3-path configuration is called a *3-odd-path configuration*. In Figure 7.1, solid lines represent edges and dotted lines represent paths with at least one edge.

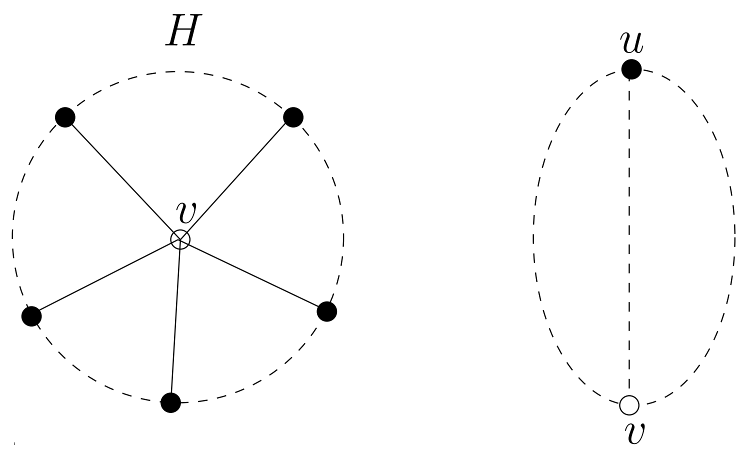

Figure 7.1: An odd wheel and a 3-odd-path configuration

Both a 3-odd-path configuration and an odd wheel have the following properties: each edge belongs to exactly two holes and the total number of edges is odd. Therefore in any signing, the sum of the labels of all holes is equal to 2 mod 4. This implies that at least one of the holes is not balanced, showing that neither 3-odd-path configurations nor odd wheels are balanceable. These are in fact the only minimal bipartite graphs that are not balanceable, as a consequence of a theorem of Truemper [194].

Theorem 7.15 (Truemper [194]) *A bipartite graph is balanceable if and only if it does not contain an odd wheel or a 3-odd-path configuration as an induced subgraph.*

7.4.1 Proof of Truemper's Theorem

In this section, we prove Theorem 7.15 following Conforti, Gerards and Kapoor [54].

For a connected bipartite graph G that contains a clique cutset K_t with t nodes, let $G'_1, \ldots, G'_n$ be the connected components of $G \backslash K_t$. The *blocks* of G are the subgraphs G_i induced by $V(G'_i) \cup K_t$ for $i = 1, \ldots, n$.

Lemma 7.16 *If a connected bipartite graph G contains a K_1 or K_2 cutset, then G is balanceable if and only if each block is balanceable.*

Proof: This follows from Camion's signing algorithm: if the cutset is a K_1 cutset, choose an arbitrary spanning tree F, and if it is a $K_2 = \{u, v\}$ cutset, let uv be in F. Then signing the blocks separately gives the same result as signing G. Since each hole of G occurs in exactly one of the blocks, the signing is balanced in each block if and only if it is balanced in G. □

So, in the remainder of the proof, we assume w.l.o.g. that G is a connected bipartite graph with no K_1 or K_2 cutset.

Lemma 7.17 *Let H be a hole of G. If $G \neq H$, then H is contained in a 3-path configuration or a wheel of G.*

Proof: Choose two nonadjacent nodes u and w in H and a uw-path $P = u, x, \ldots, z, w$ whose intermediate nodes are in $G \setminus H$ such that P is as short as possible. Such a pair of nodes u, w exists since $G \neq H$ and G has no K_1 or K_2 cutset. If $x = z$, then H is contained in a

3-path configuration or a wheel. So assume $x \neq z$. By our choice of P, u is the only neighbor of x in H and w is the only neighbor of z in H.

Let Y be the set of nodes in $V(H) - \{u, w\}$ that have a neighbor in P. If Y is empty, H is contained in a 3-path configuration. So assume Y is nonempty. By the minimality of P, the nodes of Y are pairwise adjacent and they are adjacent to u and w. This implies that Y contains a single node y and that y is adjacent to u and w. But then $V(H) \cup V(P)$ induces a wheel with center y. $\square$

For $e \in E(G)$, let G^e denote the graph with a node v_H for each hole H of G containing e and an edge $v_{H_i}v_{H_j}$ if and only if there exists a wheel or a 3-path configuration containing both holes H_i and H_j.

Lemma 7.18 *G^e is a connected graph.*

Proof: Suppose not. Let $e = uw$. Choose two holes H_1 and H_2 of G with v_{H_1} and v_{H_2} in different connected components of G^e, with the minimum distance $d(H_1, H_2)$ in $G \setminus \{u, v\}$ between $V(H_1) - \{u, w\}$ and $V(H_2) - \{u, w\}$ and, subject to this, with the smallest $|V(H_1) \cup V(H_2)|$.

Let T be a shortest path from $V(H_1) - \{u, v\}$ to $V(H_2) - \{u, v\}$ in $G \setminus \{u, v\}$. Note that T is just a node of $V(H_1) \cap V(H_2) \setminus \{u, v\}$ when this set is nonempty. The graph G' induced by the nodes in H_1, H_2 and T has no K_1 or K_2 cutset. By Lemma 7.17, H_1 is contained in a 3-path configuration or a wheel of G'. Since each edge of a 3-path configuration or a wheel belongs to two holes, there exists a hole $H_3 \neq H_1$ containing edge e in G'. Since v_{H_1} and v_{H_3} are adjacent in G^e, it follows that v_{H_2} and v_{H_3} are in different components of G^e. Furthermore, $d(H_2, H_3) = 0$. By the choice of H_1, H_2, this implies $d(H_1, H_2) = 0$ and therefore G' is induced by $V(H_1) \cup V(H_2)$. Since $H_3 \neq H_1$, $V(H_2) \cup V(H_3)$ is properly contained in $V(H_1) \cup V(H_2)$, contradicting the choice of H_1 and H_2. $\square$

Proof of Theorem 7.15: We showed already that odd wheels and 3-odd-path configurations are not balanceable. It remains to show that, conversely, if G contains no odd wheel or 3-odd-path configuration, then G is balanceable. Suppose G is a counterexample with the smallest number of nodes. By Lemma 7.16, G is connected and has no K_1 or K_2 cutset. Let $e = uv$ be an edge of G. Since $G \setminus \{u, v\}$ is connected, there exists a spanning tree F of G where u and v are leaves. Arbitrarily sign F and use Camion's signing algorithm in $G \setminus \{u\}$ and $G \setminus \{v\}$. By the minimality of G, these two graphs are balanceable and therefore Camion's algorithm yields a unique signing of all the edges except e. Furthermore, all holes not going through edge e are balanced. Since G is not balanceable, any signing of e yields some holes going through e that are balanced and some that are not. By Lemma 7.18, there exists a wheel or a 3-path configuration C containing an unbalanced hole H_1 and a balanced hole H_2 both going through edge e. Now we use the fact that each edge of C belongs to exactly two holes of C. Since the holes of C distinct from H_1 and H_2 do not go through e, they are balanced. Furthermore, applying the above fact to all edges of C, the sum of all labels in C is 1 mod 2, which implies that C has an odd number of edges. Thus C is an odd wheel or a 3-odd-path configuration, a contradiction. $\Box$

7.4.2 Another Proof of Tutte's Theorem

Truemper [194] showed how to derive Tutte's theorem from Theorem 7.15. Conforti, Gerards and Kapoor [54] give a simpler derivation based on the following result.

Lemma 7.19 *Every nonregular 0,1 matrix can be $GF(2)$-pivoted into a nonbalanceable matrix.*

Proof: Let M be a nonregular $0, 1$ matrix and let A be a smallest nonregular matrix obtained from M through a sequence of pivots and deletion of rows and columns. We will show that A is not balanceable. Suppose the contrary and let B be a balanced signing of A. By Corollary 7.14, B is minimally non totally unimodular. Pivot B on a nonzero entry, say in row i and column j, and let $\tilde{B}$ be the resulting matrix. Let $\tilde{A}$ be the matrix obtained from A by $GF(2)$-pivoting in row i and column j. $\tilde{B}$ is a signing of $\tilde{A}$. By Corollary 7.6, the matrix $\tilde{C}$ obtained from $\tilde{B}$ by removing row i and column j is minimally non totally unimodular and, by our choice of A, the corresponding submatrix C of $\tilde{A}$ is regular. By Corollary 7.14, this implies that $\tilde{C}$ is not balanced, and therefore it is an unbalanced hole matrix. The bipartite representation G of $\tilde{A}$ is a hole H plus two adjacent nodes i and j with neighbors in H. Since B is minimally non-totally unimodular, nodes i and j have even degree in G by Theorem 6.6 and Exercise 7.7. Thus i and j have an odd number of neighbors in H. Since G contains no odd wheel, i and j each have one neighbor in H. Since G is not a 3-odd-path configuration, these two neighbors are adjacent. But then A is regular, a contradiction. $\square$

Now Tutte's theorem follows from the observation that 3-odd-path configurations and odd wheels can be $GF(2)$-pivoted into graphs that contain the odd wheel (H_6, v) where H_6 is the hole of length 6 and v has degree 3. This wheel is the bipartite representation of the matrix $R(F_7)$. We leave this as an exercise.

Exercise 7.20 *Show that a 3-odd-path configuration and an odd wheel can be $GF(2)$-pivoted into a graph that contains (H_6, v) as an induced subgraph.*

Chapter 8

Decomposition by k-Sum

In this chapter, we discuss Seymour's decomposition theorem for regular matrices [186], the Tseng-Truemper decomposition theorem for binary clutters with the MFMC property [198] and results of Novick-Sebö [150] and Cornuéjols-Guenin [61] on ideal binary clutters. We adopt a matroidal point of view. See Oxley's excellent textbook [153] on matroid theory for background material.

8.1 Binary Matroids

A *matroid* M is defined by a finite ground set $V(M)$ and a family $E(M)$ of subsets of $V(M)$, called *independent sets*, with the following properties.

(i) $\emptyset \in E(M)$,

(ii) If $A \in E(M)$ and $B \subseteq A$, then $B \in E(M)$,

(iii) If $A, B \in E(M)$ and $|A| > |B|$, then there exists $a \in A \setminus B$ such that $B \cup \{a\} \in E(M)$.

A typical example is obtained by taking $V(M)$ to be a finite set of vectors over some field and $E(M)$ the subsets of linearly independent vectors.

Two matroids M_1 and M_2 are *isomorphic* if there is a bijection ψ between $V(M_1)$ and $V(M_2)$ such that A is an independent set in M_1 if and only if $\psi(A)$ is an independent set in M_2.

A *base* is a maximal independent set and a *circuit* is a minimal dependent set. By (iii), all bases have the same cardinality. A circuit that contains only one element is called a *loop*.

Example 8.1 *A matroid M is* graphic *if there exists a graph G such that $V(M)$ is the set of edges of G and $E(M)$ is the family of acyclic subsets of edges of G. It is easy to verify that the bases of M are the maximal forests of G and the circuits of M are the cycles of G.*

Exercise 8.2 *Show that, if X is an independent set in a matroid and $X \cup \{y\}$ is not, then $X \cup \{y\}$ contains a* unique *circuit.*

Exercise 8.3 *Let x, y, z be elements of a matroid M. Show that, if M has a circuit containing x, y and a circuit containing x, z, then M has a circuit containing y, z.*

In a *binary* matroid M, the elements $V(M)$ can be identified with $0, 1$ vectors, say $v_1, v_2, \ldots, v_n$, and the independent sets are the subsets of these vectors that are linearly independent over $GF(2)$. The matrix whose columns are the vectors $v_1, v_2, \ldots, v_n$ is called a *binary representation* of M. We will discuss the connection between binary matroids and binary clutters in Section 8.3.1.

Exercise 8.4 *Show that graphic matroids are binary. [Hint: Use the node/edge incidence matrix as a binary representation of M.]*

Let $X \subset V(M)$ be a base of M. The *partial representation* of M with respect to X is the $0, 1$ matrix $R(M)$ with rows indexed by the elements of X, columns indexed by the elements of $Y \equiv V(M) \setminus X$, and $r_{xy} = 1$ iff x belongs to the unique circuit contained in $X \cup \{y\}$. Note that if $R(M)$ is a partial representation of a binary matroid M, then the matrix $(I,\ R(M))$ is a binary representation of M.

Exercise 8.5 *Show that one can always go from one partial representation of a binary matroid to another by using $GF(2)$-pivoting and row* *and column permutations. [Hint: Show that pivoting on* $r_{xy} = 1$ *yields a partial representation with respect to the base* X' *obtained from* X *by exchanging* x *and* y*.]*

A binary matroid is *regular* if its partial representations are regular matrices (If one partial representation is regular, then all of them are by Exercise 8.5 and Corollary 7.8). Graphic matroids are regular (see Oxley [153] Proposition 5.1.3 for the proof).

Example 8.6 *The Fano matroid* F_7 *has the following binary representation.*

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

A partial representation of F_7 *is obtained by removing the identity matrix. It is the matrix* $R(F_7)$ *defined in the previous chapter. Therefore, the Fano matroid is not regular by Exercise 7.1.*

Let $R(M)$ be a partial representation of a binary matroid M. The transpose matrix $R(M)^T$ is a partial representation of the *dual* matroid M^*. The prefix *co* will refer to the dual matroid. For example, the circuits of M^* are called *cocircuits* of M and the dual of a graphic matroid is called a *cographic* matroid.

Example 8.7 *The dual Fano matroid* F_7^* *has the following partial representation.*

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

It is not regular (see Example 8.6).

The following characterizations of binary matroids will be useful.

Theorem 8.8 *The following statements are equivalent for a matroid* M*:*

(i) M *is binary.*

(ii) If C is a circuit and C^* is a cocircuit, then $|C \cap C^*|$ is even.

(iii) The symmetric difference of any set of circuits is a disjoint union of circuits.

(iv) If X is a base and C is a circuit, then $C = \triangle_{i \in C-X} C_i$ where C_i denotes the unique circuit in $X \cup \{i\}$.

Proof: See Oxley [153] Theorem 9.1.2. □

Let M be a binary matroid. Consider the binary representation $(I,\ R(M)^T)$ of its dual M^*. It follows from the definition of $R(M)$ that the rows of $(I,\ R(M)^T)$ are incidence vectors of circuits of M and, by Theorem 8.8(iv), every circuit C of M is of the form $C = C_1 \triangle \dots \triangle C_k$, where $C_1, \dots, C_k$ correspond to rows of $(I,\ R(M)^T)$.

Let M be a binary matroid and $R(M)$ a partial representation of M. Any submatrix of $R(M)$ is a partial representation of a binary matroid M'. A matroid M' obtained from M in this way is called a *minor* of M. For $i \in V(M)$ and a partial representation $R(M)$ where i appears as a column, the *deletion* minor $M' = M \setminus i$ is obtained by removing column i in $R(M)$. For $i \in V(M)$ and a partial representation $R(M)$ where i appears as a row, the *contraction* minor $M' = M/i$ is obtained by removing row i. Deletions, contractions and minors can be defined for general matroids (see Oxley [153]) but we need not concern ourselves with this here.

Exercise 8.9 *Let G be a graph and let $M(G)$ be the associated graphic matroid. Let M' be a minor of $M(G)$. Show that M' is a graphic matroid. Let G' be the associated graph. How is G' obtained from G?*

Tutte's theorem proved in the previous chapter (Theorem 7.10) states that a binary matroid is regular if and only if it has no minor isomorphic to F_7 or F_7^*. Tutte [202] proved a theorem with the same flavor for graphic matroids.

The graphic matroid $M(K_{3,3})$ associated with the complete bipartite graph $K_{3,3}$ has partial representation $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$. Similarly,

$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$ is a partial representation of the graphic matroid $M(K_5)$ associated with the complete graph K_5. The dual matroids $M^*(K_{3,3})$ and $M^*(K_5)$ are cographic by definition. But neither $M^*(K_{3,3})$ nor $M^*(K_5)$ is graphic.

Exercise 8.10 *Show that neither $M^*(K_{3,3})$ nor $M^*(K_5)$ is graphic.*

Theorem 8.11 (Tutte [202]) *A regular matroid is graphic if and only if it has no minor isomorphic to $M^*(K_{3,3})$ or $M^*(K_5)$.*

We refer the reader to the proof of Gerards [102].

Efficient polynomial algorithms have been developed to recognize when a binary matroid (given by a partial representation) is graphic (see, for example, Bixby and Cunningham [15]).

8.2 Decomposition of Regular Matroids

Let M be a matroid. The *rank* $r(U)$ of a set $U \subseteq V(M)$ is the maximum cardinality of an independent set contained in U. Let k be a positive integer. A *k-separation* of M is a partition (U_1, U_2) of $V(M)$ such that $|U_1| \geq k$, $|U_2| \geq k$ and $r(U_1) + r(U_2) \leq r(V(M)) + k - 1$. The matroid M is *k-connected* if it has no $(k-1)$-separation. The k-separation is *strict* if $|U_1| > k$, $|U_2| > k$.

Proposition 8.12 *A matroid is 2-connected if and only if, for every pair of elements, there is a circuit containing both.*

Proof: See Oxley [153] Proposition 4.1.4. □

Exercise 8.13 *Let G be a graph. Relate the notion of 2-connectivity in the graphic matroid associated with G to the notion of graph connectivity in G.*

Let us interpret the notion of k-separation of M in terms of its partial representations. Consider a k-separation (U_1, U_2) where the rank condition holds with equality, i.e. $r(U_1)+r(U_2) = r(V(M))+k-1$. Let X_2 be a maximal independent subset of U_2. Then enlarge X_2 by a subset X_1 of U_1 to a base of M. The partial representation matrix B of M with respect to the base $X_1 \cup X_2$ is of the form

$$B = \begin{array}{c} X_1 \\ X_2 \end{array} \begin{pmatrix} B_1 & \mathbf{0} \\ D & B_2 \end{pmatrix}$$

where the sum of the number of rows and number of columns of B_i is at least k, for $i = 1, 2$ and the rank of D over GF(2) is equal to $k - 1$.

Exercise 8.14 *Show that the sum of the number of rows and number of columns of B_i is at least k, for $i = 1, 2$ and the rank of D over GF(2) is equal to $k - 1$. [Hint: Use the binary representation.]*

When $k = 1$, then $D = \mathbf{0}$ and the matrices B_1 and B_2 are the components of a 1-*sum decomposition* of B. The reverse operation which produces B from B_1 and B_2 is called 1-*sum composition*.

Exercise 8.15 *Let $C(M)$ denote the clutter matrix whose rows are the incidence vectors of the circuits of M. Show that M has a 1-separation if and only if $C(M)$ is of the form*

$$C(M) = \begin{pmatrix} C_1 & \mathbf{0} \\ \mathbf{0} & C_2 \end{pmatrix}$$

i.e. $C(M)$ has a block diagonal structure (after permutation of rows and of columns).

When $k = 2$, B is the following matrix, where J denotes a matrix of all 1's.

$$B = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{B_1} & \mathbf{0} \\ \mathbf{0} & J & \\ \mathbf{0} & \mathbf{0} & B_2 \end{array} \right)$$

We want to extract from B two submatrices that contain B_1 and B_2, respectively, and that also contain enough information from D to reconstruct B. Clearly the knowledge of one nonzero row of D appended to B_1 and one nonzero column of D appended to B_2 will suffice. A *2-sum decomposition* of B consists of such matrices $B^1 = \begin{pmatrix} B_1 \\ \mathbf{0}\,\mathbf{1} \end{pmatrix}$ and $B^2 = \begin{pmatrix} \mathbf{1} & \\ \mathbf{0} & B_2 \end{pmatrix}$. The reverse operation which produces B from B^1 and B^2 is called 2-*sum composition.*

When $k = 3$, consider a strict 3-separation of M, say (U_1, U_2), where $|U_1|, |U_2| \geq 4$ and assume that M does not have a 1- or 2-separation. Then it can be shown that M has a partial representation $B = \begin{pmatrix} B_1 & \mathbf{0} \\ D & B_2 \end{pmatrix}$ where

- $D = \begin{pmatrix} D_1 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ D_{12} & D_2 \end{pmatrix}$ and every row of D is $\mathbf{0}$ or is identical to one of the first two rows of D or to their sum over GF(2),

- B_1 contains a row whose entries are equal to 1 in the last two columns,

- B_2 contains a column whose entries are equal to 1 in the first two rows.

A *3-sum decomposition* of B consists of

$$B^1 = \begin{pmatrix} B_1 & & \mathbf{0} \\ D_1 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} 1 \\ 1 \end{matrix} \end{pmatrix} \quad \text{and} \quad B^2 = \begin{pmatrix} 1 \;\; 1 & \mathbf{0} \\ \begin{matrix} 1 & 0 \\ 0 & 1 \\ & D_2 \end{matrix} & B_2 \end{pmatrix}.$$

The reverse operation which produces B from B^1 and B^2 is called 3-*sum composition.*

Exercise 8.16 *Any 1-, 2-, or 3-sum composition of two regular matroids is regular.*

Seymour [186] showed that a converse exists, namely any regular matroid can be 1-, 2-, or 3-sum decomposed into graphic and cographic matroids and copies of a 10-element matroid called R_{10}. This binary matroid R_{10} has two distinct partial representations (up to permutation of rows and columns):

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Exercise 8.17 *Show that R_{10} is a regular matroid.*

Theorem 8.18 (Seymour [186]) *Every regular matroid can be decomposed into graphic matroids, cographic matroids and matroids isomorphic to R_{10} by repeated 1-, 2- and 3-sum decompositions.*

8.2.1 Proof Outline

An important ingredient in Seymour's proof of Theorem 8.18 is the Splitter Theorem [186]. Let C be a class of binary matroids closed under isomorphism and the taking of minors. A 3-connected matroid $M \in C$ with at least six elements is a *splitter* of C if every matroid $N \in C$ containing M as a proper minor has a 2-separation. A *single-element extension* of a matroid M is the inverse operation of deletion, namely it is a matroid N with elements $V(M) \cup \{e\}$ such that M is obtained from N by deleting e. If N^* is a single-element extension of M^*, then N is called a *single-element coextension* of M. The graph consisting of a chordless cycle with k nodes and an additional node adjacent to all k nodes of the cycle is the *wheel* W_k.

Theorem 8.19 (Splitter Theorem) *Let C be a class of binary matroids closed under isomorphism and the taking of minors. Let M be a 3-connected matroid in C with at least six elements. Assume M is not the graphic matroid of a wheel W_k. Then M is a splitter of C if and only if C does not contain a 3-connected single-element extension or coextension of M.*

Using the splitter theorem, it is routine to verify the following result (recall the definitions at the end of Section 8.1).

Theorem 8.20 *The graphic matroid $M(K_5)$ is a splitter of the regular matroids with no minor isomorphic to $M(K_{3,3})$.*

A certain 12-element matroid R_{12} plays a central role in Seymour's proof of Theorem 8.18. A partial representation of R_{12} is

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Theorem 8.21 *Let M be a 3-connected regular matroid with a minor isomorphic to $M(K_{3,3})$. Assume that M is neither graphic nor cographic but that every proper minor of M is graphic or cographic. Then M is isomorphic to R_{10} or R_{12}.*

Proof: See [186] or [195] for this proof. □

Theorems 8.11, 8.20 and 8.21 imply the following result.

Theorem 8.22 *A 3-connected regular matroid is graphic or cographic if and only if it has no minor isomorphic to R_{10} or R_{12}.*

Proof: The necessity follows by checking that R_{10} and R_{12} are neither graphic nor cographic. To prove the sufficiency, assume M is 3-connected regular but not graphic or cographic. Since M is not cographic, M has a minor isomorphic to $M(K_5)$ or $M(K_{3,3})$, by Tutte's theorem (Theorem 8.11). This implies that M must have a $M(K_{3,3})$ minor, since otherwise M has a 2-separation by Theorem 8.20, contradicting the assumption that M is 3-connected. Now, by Theorem 8.21, M contains a minor isomorphic to R_{10} or R_{12}. □

Theorem 8.23 *The matroid R_{10} is a splitter of the class of regular matroids.*

Proof: A routine application of Theorem 8.19. □

Theorem 8.24 *Let M be a regular matroid with an R_{12} minor. Then R_{12} has a* 3*-separation that induces a 3-separation of M.*

Proof: See [186] or [195] for this proof. □

Theorems 8.22, 8.23 and 8.24 imply Seymour's decomposition theorem for regular matroids (Theorem 8.18).

We close this section with another consequence of the Splitter Theorem (see Oxley [153] Proposition 11.2.3 for a proof).

Theorem 8.25 (Seymour [187]) *Let M be a 3-connected binary matroid with no F_7^* minor. Then M is regular or $M = F_7$.*

8.2.2 Recognition Algorithm

Theorem 8.18 can be used to recognize in polynomial time whether a 0,1 matrix is regular. Indeed, 1-, 2- and 3-separations can be found in polynomial time (see Truemper [195] Section 8.4), one can show that the total number of matrices generated by repeated 1-, 2- and 3-sum decompositions is linear, and R_{10}, graphic and cographic matrices can be recognized in polynomial time (see, for example, Truemper [195] Section 10.6). Thus regularity can be checked in polynomial time. Therefore, through Camion's signing algorithm, one can check in polynomial time whether a $0, \pm 1$ matrix is totally unimodular. See also Chapter 20 in Schrijver's book [173]. Note that no polynomial recognition algorithm for total unimodularity was known before Seymour's decomposition approach.

8.3 Binary Clutters

8.3.1 Relation to Binary Matroids

Let M be a binary matroid and $S \subseteq V(M)$ a subset of its elements. The pair (M, S) is called a *signed matroid*, and S is called the *signature* of M. We say that a circuit C of M is *odd* (resp. *even*) if $|C \cap S|$ is odd (resp. even).

Proposition 8.26 *The odd circuits of a signed matroid form a binary clutter.*

Proof: Consider a signed matroid (M, S) and let C_1, C_2, C_3 be three odd circuits. Since S intersects each of C_1, C_2, C_3 with odd parity, so does $L = C_1 \triangle C_2 \triangle C_3$. By Theorem 8.8(iii), L is a disjoint union of circuits. One of these circuits must be odd since $|L \cap S|$ is odd. The result now follows from Proposition 5.15. $\square$

Let M be a binary matroid. Any nontrivial binary clutter obtained as the odd circuit clutter of the signed matroid (M, S), for some S, is called a *source* of M. Any nontrivial binary clutter $\mathcal{H}$ such that every circuit of M is of the form $T_1 \triangle T_2$, for $T_1, T_2 \in E(\mathcal{H})$, is called a *lift* of M. One can show that a lift of M is the blocker of a source of M^*.

Exercise 8.27 *Show that the binary matroid F_7 has three sources and one lift.*

Exercise 8.28 *Show that $b(\mathcal{O}_{K_5})$ is a source of the binary matroid R_{10}.*

In a binary matroid, any circuit C and cocircuit D have an even intersection (Theorem 8.8(ii)). So, if D is a cocircuit, then (M, S) and $(M, S \triangle D)$ have exactly the same odd circuits.

Remark 8.29 *Let (M, S) be a signed matroid and $\mathcal{H}$ the clutter of its odd circuits.*

- *$\mathcal{H} \setminus e$ is the clutter of odd circuits of the signed matroid $(M \setminus e,\ S - \{e\})$.*
- *If $e \notin S$, then $\mathcal{H}/e$ is the clutter of odd circuits of the signed matroid $(M/e,\ S)$.*
- *If $e \in S$ is not a loop of M, then $\mathcal{H}/e$ is the clutter of odd circuits of the signed matroid $(M/e,\ S \triangle D)$ where D is any cocircuit containing e.*
- *If $e \in S$ is a loop of M, then $\mathcal{H}/e$ is a trivial clutter.*

Given a nontrivial binary clutter $\mathcal{H}$, the minimal sets in $E(\mathcal{H}) \cup \{T_1 \triangle T_2 : T_1, T_2 \in E(\mathcal{H})\}$ form the circuits of a binary matroid $u(\mathcal{H})$. This binary matroid is called the *up matroid* of $\mathcal{H}$. Since $\mathcal{H}$ is binary, the minimal transversals of $\mathcal{H}$ intersect with odd parity exactly the circuits of $u(\mathcal{H})$ that are edges of $\mathcal{H}$. It follows that $\mathcal{H}$ is the clutter of odd circuits of the signed matroid $(u(\mathcal{H}), S)$ where S is any minimal transversal of $\mathcal{H}$. Moreover, this representation is essentially unique (see for example [61]):

Proposition 8.30 *Let (M, S) and (M', S') be signed matroids that have the same clutter of odd circuits $\mathcal{H}$. If M is a 2-connected matroid and $\mathcal{H}$ is a nontrivial clutter, then $M = M' = u(\mathcal{H})$.*

To prove this, we use the following result of Lehman [131] (see Oxley [153] Theorem 4.3.2 or Exercise 9 of Section 9.3).

Theorem 8.31 (Lehman [131]) *Let t be an element of a 2-connected binary matroid M. The circuits of M not containing t are of the form $C_1 \triangle C_2$ where C_1 and C_2 are circuits of M containing t.*

Proof of Proposition 8.30: Let N be the binary matroid with elements $V(M) \cup \{t\}$ and circuits $\Gamma = C$ when C is an even circuit of (M, S) and $\Gamma = C \cup \{t\}$ when C is an odd circuit of (M, S). Define N' similarly from (M', S'). Since $\mathcal{H}$ is nontrivial, at least one circuit of N contains t and some $x \neq t$. Since M is 2-connected, for every pair of elements in $V(M)$, there is a circuit of M containing both by Proposition 8.12. So for x and any $v \in V(N)$, there is a circuit of N containing both. By Exercise 8.3 it follows that, for any pair of elements in $V(N)$, there is a circuit containing both. So N is 2-connected by Proposition 8.12. Furthermore, every $v \in V(\mathcal{H})$ belongs to an edge of $\mathcal{H}$. So N' is 2-connected as well. By Theorem 8.31, a 2-connected matroid is uniquely determined by the set of circuits containing any fixed element. In particular, N and N' are uniquely determined by the circuits containing t. This implies $N = N'$. Since $M = N/t$ and $M' = N'/t$, it follows that $M = M' = u(\mathcal{H})$. $\square$

Proposition 8.32 (Novick and Sebö [150])

A binary clutter $\mathcal{H}$ is the odd cycle clutter of a signed graph if and only if $u(\mathcal{H})$ is a graphic matroid.

A binary clutter $\mathcal{H}$ is the T-cut clutter of a graft if and only if $u(\mathcal{H})$ is a cographic matroid.

The next result relates the minors of the matroid $u(\mathcal{H})$ to the minors of the clutter $\mathcal{H}$. For a clutter $\mathcal{H}$ and $v \notin V(\mathcal{H})$, the clutter $\mathcal{H}^+$ has vertex set $V(\mathcal{H}) \cup \{v\}$ and edge set $\{A \cup \{v\} : A \in E(\mathcal{H})\}$.

Theorem 8.33 (Cornuéjols-Guenin [61]) *Let $\mathcal{H}$ be a nontrivial binary clutter such that its up matroid $u(\mathcal{H})$ is 2-connected, and let N be a 2-connected binary matroid. Then $u(\mathcal{H})$ has N as a minor if and only if $\mathcal{H}$ has $\mathcal{H}_1$ or $\mathcal{H}_2^+$ as a minor, where $\mathcal{H}_1$ is a source of N and $\mathcal{H}_2$ is a lift of N.*

To prove this, we use the following result of Brylawski [20] and Seymour [182] (see Oxley [153] Proposition 4.3.6).

Theorem 8.34 (Brylawski [20], Seymour [182]) *Let M be a 2-connected matroid and N a 2-connected minor of M. For any $i \in V(M) - V(N)$, at least one of $M \setminus i$ or M/i is 2-connected and has N as a minor.*

Proof of Theorem 8.33: $\mathcal{H}$ is the clutter of odd circuits of the signed matroid (M, S) where $M = u(\mathcal{H})$ and S is a minimal transversal of $\mathcal{H}$.

Suppose first that $\mathcal{H}$ has a minor $\mathcal{H}_1$ that is a source of N. Then $\mathcal{H}_1$ is nontrivial and it follows from Remark 8.29 that $\mathcal{H}_1$ is the clutter of odd circuits of a signed matroid (N', S') where N' is a minor of M. Since $\mathcal{H}_1$ is nontrivial and N is 2-connected, $N = N' = u(\mathcal{H}_1)$ by Proposition 8.30. So N is a minor of M.

Suppose now that $\mathcal{H}$ has a minor $\mathcal{H}_2^+$ where $\mathcal{H}_2$ is a lift of N. Let t be the vertex of $V(\mathcal{H}_2^+) - V(\mathcal{H}_2)$. Since $\mathcal{H}_2^+$ is a nontrivial minor of $\mathcal{H}$, it is the clutter of odd circuits of a signed matroid (N', S') where N' is a minor of M. Since $u(\mathcal{H}_2^+)$ is 2-connected, $N' = u(\mathcal{H}_2^+)$ by Proposition 8.30. So N' is 2-connected. Therefore, by Theorem 8.31 and the definition of lift, $N = N' \setminus t$. So N is a minor of M.

Now we prove the converse. Suppose that M has N as minor and does not satisfy the theorem. Let $\mathcal{H}$ be such a counterexample with smallest number of vertices. Clearly, N is a proper minor of M as otherwise $u(\mathcal{H}) = N$, i.e. $\mathcal{H}$ is a source of N. By Theorem 8.34, for every $i \in V(M) - V(N)$, one of $M \setminus i$ and M/i is 2-connected and has N as a minor. Suppose first that M/i is 2-connected and has an N minor. Since M is 2-connected, i is not a loop of M and therefore $\mathcal{H}/i$ is nontrivial by Remark 8.29, a contradiction to the choice of $\mathcal{H}$ with smallest number of vertices. Thus, for every $i \in V(M) - V(N)$, $M \setminus i$ is 2-connected and has an N minor. By minimality, $\mathcal{H} \setminus i$ must be trivial. It follows from Remark 8.29 that all odd circuits of (M, S) use i. As $M = u(\mathcal{H})$, even circuits of M do not use i.

We claim that $V(M) - V(N) = \{i\}$. Suppose not and let $j \neq i$ be an element of $V(M) - V(N)$. The set of circuits of (M, S) using j is exactly the set of odd circuits. It follows that the elements i, j must be in series in M. But then $M \setminus i$ is not connected, a contradiction.

Therefore $V(M) - V(N) = \{i\}$ and $M \setminus i = N$. As the circuits of (M, S) using i are exactly the odd circuits of (M, S), it follows that column i of $\mathcal{H}$ consists of all 1's, i.e. $\mathcal{H} = \mathcal{H}_2^+$. By Theorem 8.31 applied to i and M, every circuit of N is of the form $T_1 \triangle T_2$ where $T_1, T_2 \in E(\mathcal{H}_2)$. So $\mathcal{H}_2$ is a lift of N. $\Box$

There is another important connection between binary matroids and binary clutters. Given a matroid M and an element ℓ of the matroid, *port* $\ell(M)$ is the clutter whose vertices are the elements of the matroid distinct from ℓ and whose edges are the sets $C - \{\ell\}$ where C is a circuit of M containing ℓ. If M is a binary matroid, then $\ell(M)$ is a binary clutter. Conversely, every binary clutter can be obtained as the port of a binary matroid (Lehman [131]).

Exercise 8.35 *Let G be a graph and let $\ell = st$ be one of its edges. If M denotes the graphic matroid associated with G, describe the port $\ell(M)$.*

Exercise 8.36 *Let M be a binary matroid such that its port $\ell(M)$ is a nontrivial clutter. Show that $\ell(M)$ is a source of M/ℓ and a lift of $M \setminus \ell$.*

8.3.2 The MFMC Property

This section can be viewed as an extension of Seymour's decomposition theory for regular matroids.

Tutte [202] showed that a binary matroid is regular if and only if it does not have a minor isomorphic to F_7 or F_7^* (Theorem 7.10), and Seymour [186] showed that a regular matroid has a k-separation, for $k \leq 3$, or is graphic, cographic or R_{10} (Theorem 8.18).

Exercise 8.37 *Show that ports of regular matroids have the MFMC property.*

Tseng and Truemper extend Seymour's k-separation theorem to nonregular matroids that have a port with the MFMC property.

Theorem 8.38 (Tseng-Truemper [198]) *Let M be a 3-connected binary nonregular matroid with distinguished element ℓ. Assume $\ell(M)$ has the MFMC property and $M \neq F_7$. Then M has a 3-separation.*

Bixby and Rajan [16] give a shorter proof of this theorem.

8.3.3 Idealness

A binary clutter $\mathcal{H}$ has a *k-separation* if $u(\mathcal{H})$ has a k-separation, i.e. there exists a partition (U_1, U_2) of $V(\mathcal{H})$ such that $|U_1| \geq k$, $|U_2| \geq k$ and $r(U_1) + r(U_2) \leq r(V(\mathcal{H})) + k - 1$. The k-separation is *strict* if $|U_1| > k$, $|U_2| > k$. The binary clutter $\mathcal{H}$ is *k-connected* if it has no $(k-1)$-separation. It is *internally k-connected* if it has no strict $(k-1)$-separation.

Theorem 8.39 [61] *Minimally nonideal binary clutters are 3-connected.*

Exercise 8.40 *Show that the minimally nonideal binary clutter F_7 is not 4-connected.*

Theorem 8.41 [61] *Minimally nonideal binary clutters are internally 4-connected.*

Conjecture 8.42 *Minimally nonideal binary clutters are internally 5-connected.*

Let Q_6 be the clutter where $V(Q_6)$ is the set of edges of K_4 and $E(Q_6)$ the set of triangles of K_4. The next result proves Seymour's conjecture (Conjecture 5.26) for the class of clutters that do not have $\mathcal{Q}_6^+$ or $b(\mathcal{Q}_6)^+$ as a minor.

Theorem 8.43 (Cornuéjols-Guenin [61]) *A binary clutter is ideal if it does not have $\mathcal{F}_7$, $\mathcal{O}_{K_5}$, $b(\mathcal{O}_{K_5})$, $\mathcal{Q}_6^+$, or $b(\mathcal{Q}_6)^+$ as a minor.*

Proof: It suffices to show that every *mni* clutter $\mathcal{H}$ contains one of the minors in the statement of the theorem.

Claim 1: The result holds if $u(\mathcal{H})$ has no F_7^* minor.

Proof of Claim 1: When $u(\mathcal{H}) = R_{10}$, then $\mathcal{H}$ is one of the sources of R_{10}. We leave it as an exercise to show that R_{10} has 6 sources. One such source is $b(\mathcal{O}_{K_5})$ (see Exercise 8.28) and the other five are ideal.

When $u(\mathcal{H})$ is graphic, then $\mathcal{H}$ is ideal if and only if $\mathcal{H}$ has no $\mathcal{O}_{K_5}$ minor, by Proposition 8.32 of Novick-Sebö and Guenin's theorem (Theorem 5.6).

When $u(\mathcal{H})$ is cographic, then $\mathcal{H}$ is ideal, by Proposition 8.32 of Novick-Sebö and the Edmonds-Johnson theorem (Theorem 2.1).

By the connectivity results (Theorems 8.39 and 8.41), $u(\mathcal{H})$ is 3-connected and internally 4-connected. So, by Seymour's theorem (Theorem 8.18), the result holds when $u(\mathcal{H})$ is a regular matroid.

Now consider the case when $u(\mathcal{H})$ is not regular. Another theorem of Seymour (Theorem 8.25) shows that $u(\mathcal{H}) = F_7$. So $\mathcal{H}$ is a source of F_7. It was shown in Exercise 8.27 that F_7 has three sources. Two of these sources are ideal and the third is the clutter $\mathcal{F}_7$. So the result holds.

Claim 2: The result holds if $u(\mathcal{H})$ has an F_7^* minor.

Proof of Claim 2: By Theorem 8.33, $u(\mathcal{H})$ has an F_7^* minor if and only if $\mathcal{H}$ has $\mathcal{H}_1$ or $\mathcal{H}_2^+$ as a minor, where $\mathcal{H}_1$ is a source of F_7^* and $\mathcal{H}_2$ is a lift of F_7^*. It follows from Exercise 8.27 that F_7^* has one source and three lifts. The source is Q_6^+, which is one of the excluded minors in the statement of the theorem. For the three lifts $\mathcal{H}_2$ of F_7^*, one can check that $\mathcal{H}_2^+$ contains $\mathcal{F}_7$, $\mathcal{Q}_6^+$ and $b(\mathcal{Q}_6)^+$ as minors, respectively, which are excluded minors in the statement of the theorem. $\square$

The class of clutters of T-cuts is closed under minor taking. Moreover, it is not hard to check that none of the five excluded minors of Theorem 8.43 are clutters of T-cuts. Thus Theorem 8.43 implies that clutters of T-cuts are ideal, and thus that their blocker, the clutters of T-joins are ideal. Hence Theorem 8.43 implies the Edmonds-Johnson theorem (Theorem 2.1). Similarly, the class of clutters of odd circuits is closed under minor taking. Moreover, it can be shown that $\mathcal{O}_{K_5}$ is the only clutter of odd circuits among the five excluded minors. It follows that Theorem 8.43 also implies Guenin's theorem (Theorem 5.6). Note, however, that the proof of Theorem 8.43 uses these two results.

Chapter 9

Decomposition of Balanced Matrices

In this chapter, we present a polynomial time recognition algorithm for balanced 0,1 matrices. By contrast, no polynomial recognition algorithm is known for perfection or for idealness. The algorithm decomposes a balanced matrix into totally unimodular matrices. The decomposition result is best explained in terms of graphs. Given a $0,1$ matrix A, the *bipartite representation of* A is the bipartite graph $G = (V^r \cup V^c, E)$ having a node in V^r for every row of A, a node in V^c for every column of A and an edge ij joining nodes $i \in V^r$ and $j \in V^c$ if and only if the entry a_{ij} of A equals 1. If A is balanced (totally unimodular resp.), the bipartite graph G is said to be balanced (totally unimodular resp.).

A *double star* is a tree where at most two nodes have degree greater than one. A node set S is a *double star cutset* of G if $G \setminus S$ has more connected components than G and S induces a double star.

Theorem 9.1 (Conforti, Cornuéjols and Rao [51]) *If a bipartite graph is balanced but not totally unimodular, then it has a double star cutset.*

This theorem can be used to recognize whether a bipartite graph G is balanced. The recognition algorithm recursively decomposes the graph until no double star cutset exists. Then each of the final blocks is checked for total unimodularity. For this approach to work in polynomial time, we need three properties: (i) the presence of a double star cutset should be detectable in polytime, (ii) the blocks of the decomposition should be balanced if and only if the original graph is balanced, and (iii) the total number of blocks generated in the algorithm should be polynomial. We discuss these issues in Section 9.1. Our exposition follows [51].

In Section 9.2, we outline the proof of Theorem 9.1. Finally, in Section 9.3, we present the generalization of this work to balanced $0, \pm 1$ matrices by Conforti, Cornuéjols, Kapoor and Vušković [49].

9.1 Recognition Algorithm for Balanced 0,1 Matrices

Given a connected bipartite graph G, let S be a node set such that $G \setminus S$ is disconnected. Let $G'_1, \ldots, G'_k$ denote the connected components of $G \setminus S$. The *blocks* of the decomposition of G by S are the graphs G_i induced by $V(G'_i) \cup S$.

A difficulty with double star cutsets is that this natural definition of blocks does not satisfy Property (ii) stated in the introduction: unbalanced holes can be broken during the decomposition (give an example!). To deal with this problem, the first step of the algorithm is a preprocessing step, referred to as "cleaning", guaranteeing that a smallest unbalanced hole (if one exists) is never broken. This cleaning step relies on properties studied in the next section.

9.1.1 Smallest Unbalanced Holes

Let G be a bipartite graph that is not balanced and let H^* be a smallest unbalanced hole in G. We first study properties of nodes $u \in V(G) - V(H^*)$ with more than one neighbor in $V(H^*)$. Such nodes are said to be *strongly adjacent* to H^*. Node u is *odd-strongly adjacent* if u has an odd number greater than one of neighbors in H^*, and it is *even-strongly adjacent* if it has an even number greater than one of neighbors in H^*.

Let $A^r(H^*)$ and $A^c(H^*)$ contain the odd-strongly adjacent nodes to H^* that belong to V^r and V^c respectively. The following properties were proven by Conforti and Rao in [58].

Property 9.2 *There exists a node $x^r \in V^r \cap V(H^*)$ adjacent to all the nodes in $A^c(H^*)$. There exists a node $x^c \in V^c \cap V(H^*)$ adjacent to all the nodes in $A^r(H^*)$.*

Exercise 9.3 *Prove Property 9.2*

Two nodes of G that have exactly the same neighbors in a subgraph K of G are said to be *twins with respect to K*. When K is clear from the context, such nodes are simply said to be *twins*.

Property 9.4 *Every even-strongly adjacent node to H^* is a twin of a node in H^**

Exercise 9.5 *Prove Property 9.4*

The above properties were used in [59] to design a polytime algorithm to test whether a linear bipartite graph is balanced (a bipartite graph is *linear* if it contains no 4-cycle). To test balancedness in general, we need the following additional properties of strongly adjacent nodes of H^*. Let $N(v)$ denote the set of neighbors of v.

Definition 9.6 *A* tent $\tau(H, u, v)$ *is a subgraph of G induced by a hole H and two adjacent nodes u and v that are even-strongly adjacent to H with the following property:*

The nodes of H can be partitioned into two subpaths containing the nodes in $N(u) \cap V(H)$ and $N(v) \cap V(H)$ respectively.

A tent $\tau(H, u, v)$ is referred to as a *tent containing H*. We now study properties of a tent $\tau(H^*, u, v)$ containing a smallest unbalanced hole H^*. By Property 9.4, both u and v are twins of nodes of H. We assume throughout that the first node, say u, in the definition of a tent $\tau(H, u, v)$ belongs to V^r and that the second node, say v, belongs to V^c. We use the notation of Figure 9.1, where nodes $u_1, u_0, u_2, v_1, v_0, v_2$ are encountered in this order when traversing H^*.

Lemma 9.7 *Let H^* be a smallest unbalanced hole and $\tau(H^*, u, v)$ be a tent containing it. At least one of the two sets $N(v_0) \cup N(u_1)$, $N(v_0) \cup N(u_2)$ contains $A^r(H^*)$. At least one of the two sets $N(u_0) \cup N(v_1)$, $N(u_0) \cup N(v_2)$ contains $A^c(H^*)$.*

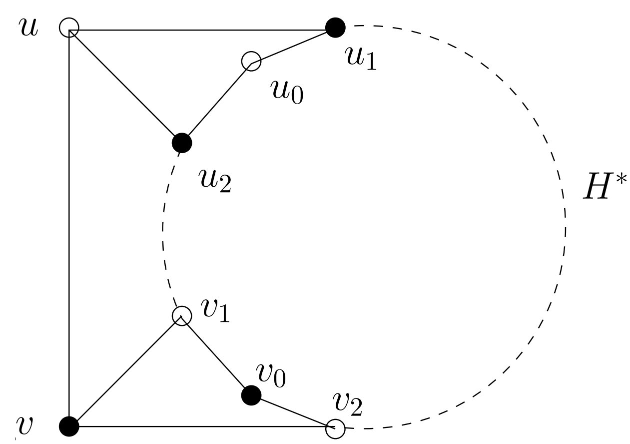

Figure 9.1: Tent

Proof: By symmetry, we only need to prove the first statement. Suppose v_0 is not adjacent to a node $w \in A^r(H^*)$. Consider the hole H_1^* obtained from H^* by replacing v_0 with node v. Now w is not adjacent to v, for otherwise w is even-strongly adjacent to H_1^*, violating Property 9.4. Therefore, w is in $A^r(H_1^*)$. Node u is in $A^r(H_1^*)$ and has neighbors u_1, u_2 and v in H_1^*. By Property 9.2, all nodes in $A^r(H_1^*)$ have a common neighbor in H_1^*. So it follows that this common neighbor must be u_1 or u_2. □

Lemma 9.8 *Let H^* be a smallest unbalanced hole and $\tau(H^*, u, v)$, $\tau(H^*, w, y)$ two tents containing H^*, where w_1, w_2 are the neighbors of w and y_1, y_2 are the neighbors of y in H^*. Let w_0 and y_0 be the common neighbors in H^* of w_1, w_2 and y_1, y_2 respectively. Then at least one of the following properties holds:*

- *Nodes u_1 and u_2 coincide with w_1 and w_2.*
- *Nodes v_1 and v_2 coincide with y_1 and y_2.*
- *Nodes u_0 and y are adjacent.*
- *Nodes v_0 and w are adjacent.*

Proof: Suppose the contrary. Then u, v, w, y are all distinct nodes and one of the following two cases occurs. The edges of H^* can be partitioned in two paths P_1, P_2 with common endnodes so that either (Case 1:) P_1 contains u_1, u_2, v_1, v_2 and P_2 contains w_1, w_2, y_1, y_2 or (Case 2:) P_1 contains u_1, u_2, y_1, y_2 and P_2 contains v_1, v_2, w_1, w_2.

Suppose u and y are adjacent and consider the hole H^*_{wy} contained in $V(H^*) \cup \{w, y\}$, containing w, y, u_1, u_2. Then (H^*_{wy}, u) is an odd wheel (recall the definition given in Section 7.4) and all the holes of (H^*_{wy}, u) are smaller than H^*. Since one of them is unbalanced, we have a contradiction to the minimality of H^*. By symmetry, w and v are nonadjacent as well.

In Case 1, consider the hole H^*_{vwy} contained in $V(H^*) \cup \{v, w, y\}$, containing v, w, y, u_1, u_2. Then (H^*_{vwy}, u) is an odd wheel and all the holes of (H^*_{vwy}, u) are smaller than H^*, a contradiction. In Case 2, nodes u and y are connected by a 3-odd-path configuration. The three holes in this 3-odd-path configuration are smaller than H^* and at least one of them is unbalanced. $\Box$

9.1.2 Recognition Algorithm

In this section, we give an algorithm to test whether a bipartite graph is balanced.

Definition 9.9 *A hole H is said to be* clean in G *if the following three conditions hold:*

(i) *No node is odd-strongly adjacent to H.*

(ii) *Every even-strongly adjacent node is a twin of a node in H.*

(iii) *There is no tent containing H.*

A *wheel* (H, v) is a hole H together with a node $v \notin V(H)$ with at least three neighbors in H. A subpath of H having two nodes of $N(v) \cap V(H)$ as endnodes and only nodes of $V(H) - N(v)$ as intermediate nodes is called a *sector* of (H, v). A *short 3-wheel* is a wheel with three sectors, at least two of which have length 2.

A node u in a bipartite graph G is said to be *dominated* if there exists a node v, distinct from u, such that $N(u) \subseteq N(v)$. A bipartite graph G is said to be *undominated* if G contains no dominated nodes.

RECOGNITION ALGORITHM

Input: A bipartite graph G.
Output: G is identified as balanced or not balanced.

Step 1 Apply Procedure 1 to check whether G contains a short 3-wheel. If so, G is not balanced, otherwise go to Step 2.

Step 2 Apply Procedure 2 to create at most $|V^r|^4|V^c|^4$ induced subgraphs $G_1, \ldots, G_p$ such that, if G is not balanced, at least one of the induced subgraphs created, say G_t, contains an unbalanced hole H^* that is clean in G_t.

Step 3 Apply Procedure 3 to each of the graphs $G_1, \ldots, G_p$ to decompose them into undominated induced subgraphs $B_1, \ldots, B_s$ that do not contain a double star cutset. While decomposing a graph with a double star cutset $N(u) \cup N(v)$, Procedure 3 also checks for the existence of a 3-odd-path configuration containing nodes u and v and nodes in two distinct connected components resulting from the decomposition. If such a configuration is found, then G is not balanced, otherwise go to Step 4.

Step 4 Test whether all the blocks $B_1, \ldots, B_s$ are totally unimodular. If so, G is balanced, otherwise G is not balanced.

An algorithm to test whether a bipartite graph is totally unimodular can be found in [173]. See Section 8.2.2 for a brief description. In the remainder of this section, we describe the procedures used in Steps 1 to 3.

Short 3-Wheels

PROCEDURE 1, for identifying whether G contains a short 3-wheel, can be described as follows: Let $C = a_1, a_2, a_3, a_4, a_5, a_6, a_1$ be a 6-cycle of G having unique chord a_2a_5. If a_1 and a_3 are in the same connected component of $G \setminus (N(a_2) \cup N(a_4) \cup N(a_5) \cup N(a_6) - \{a_1, a_3\})$, or if a_4 and a_6 are in the same connected component of $G \setminus (N(a_1) \cup N(a_2) \cup N(a_3) \cup N(a_5) - \{a_4, a_6\})$, then a short 3-wheel containing C is identified.

Otherwise G has no short 3-wheel containing C. Perform such a test for all 6-cycles of G with a unique chord.

The complexity of this procedure is of order $O(|V^r|^4|V^c|^4)$.

Clean Unbalanced Holes

Next we show how to create at most $|V^r|^4|V^c|^4$ induced subgraphs of G such that, if G is not balanced, one of these subgraphs, say G_t, contains a smallest unbalanced hole of G that is clean in G_t.

Given a graph F and nodes i, j, k, l of F that induce the chordless path i, j, k, l, we define F_{ijkl} to be the induced subgraph obtained from F by removing the nodes in $N(j) \cup N(k) - \{i, j, k, l\}$.

PROCEDURE 2

Input: A bipartite graph G.

Output: A family $\mathcal{L} = G_1, \ldots, G_p$, where $p \leq |V^r|^4|V^c|^4$, of induced subgraphs of G such that if G is not balanced, then a smallest unbalanced hole H^* in G appears in one of the subgraphs $G_t \in \mathcal{L}$ and H^* is clean in G_t.

Step 1 Let $\mathcal{L}^* = \{G_{ijkl}$: nodes i, j, k, l of G induce the chordless path $i, j, k, l\}$

Step 2 Let $\mathcal{L} = \{Q_{wxyz}$: the graph Q is in $\mathcal{L}^*$ and nodes w, x, y, z of Q induce the chordless path $w, x, y, z\}$.

We now prove the validity of Procedure 2.

Lemma 9.10 *If G is not balanced, then a smallest unbalanced hole H^* in G appears in one of the subgraphs $G_t \in \mathcal{L}$ and H^* is clean in G_t.*

Proof: Let H^* be any smallest unbalanced hole in G. Choose two induced paths u_1, u_0, u_2, u_3 and v_1, v_0, v_2, v_3 on H^* as follows:

- If no tent contains H^*: $A^r(H^*) \subseteq N(u_0)$ and $A^c(H^*) \subseteq N(v_0)$. (This choice is possible by Property 9.2.)
- If some tent $\tau(H^*, u, v)$ contains H^*: $\{u_1, u_2\} = N(u) \cap V(H^*)$ and $\{v_1, v_2\} = N(v) \cap V(H^*)$. By Lemma 9.7, we can index u_i, $i = 1, 2$, so that $A^r(H^*) \subseteq N(v_0) \cup N(u_2)$ and we can index v_i,

$i = 1, 2$, so that $A^c(H^*) \subseteq N(u_0) \cup N(v_2)$. By Lemma 9.8, for every tent $\tau(H^*, w, y)$ containing H^*, w or y is adjacent to one of the nodes in $\{u_0, u_2, v_0, v_2\}$.

So $(G_{u_1u_0u_2u_3})_{v_1v_0v_2v_3}$ belongs to $\mathcal{L}$, it contains H^*, it has no tent that contains H^*, and H^* has no odd strongly adjacent node. Furthermore, by Proposition 9.4, the even-strongly adjacent node to H^* are twins of nodes in H^*. □

Double Star Cutset Decompositions

We describe a procedure to decompose a bipartite graph G into blocks that are induced subgraphs of G and do not have a double star cutset.

Definition 9.11 *Let H be a hole in a graph. Then $\mathcal{C}(H) = \{H_i \mid H_i$ is a hole that can be obtained from H by a sequence of holes $H = H_0, H_1, \ldots, H_i$, where $|V(H_j) - V(H_{j-1})| = 1$, for $j = 1, 2, \ldots, i\}$.*

Lemma 9.12 *Let G be a bipartite graph that is not balanced and contains no short 3-wheel. If H is a clean smallest unbalanced hole in G, then every hole H_i in $\mathcal{C}(H)$ is clean in G, $|H_i| = |H|$ and $\mathcal{C}(H_i) = \mathcal{C}(H)$.*

Proof: Let $H = H_0, H_1, \ldots, H_i$ be a sequence of holes as in Definition 9.11. It suffices to show the lemma for H_1. Since $A^r(H) \cup A^c(H) = \emptyset$, by Property 9.4, H_1 has been obtained from $H = x_1, x_2, x_3, \ldots, x_n, x_1$ by substituting one node, say x_3, with its twin y_3. We assume w.l.o.g. that $x_3, y_3 \in V^r$. So $|H_1| = |H|$ and $\mathcal{C}(H_1) = \mathcal{C}(H)$.

Assume $A^r(H_1) \cup A^c(H_1) \neq \emptyset$. Since H is clean, then H must contain a twin y_i of x_i in V^c, where y_i is adjacent to y_3 but not to x_3. Now $\tau(H, y_3, y_i)$ is a tent, a contradiction to the assumption that H is clean, and this proves that H_1 satisfies (i) of Definition 9.9.

The fact that H is clean shows that H_1 satisfies (ii) of Definition 9.9.

Finally, assume H_1 contains a tent $\tau(H_1, x, y)$, where $x \in V^r$. Then $x_3 \neq x$, else $y \in A^r(H)$. So x is a twin of a node in H and y is adjacent to y_3. We can assume that x_1 is the unique neighbor of y in H. Now let x_i be the neighbor of x with lowest index and let $C = x_1, x_2, x_3 \ldots, x_i, x, y, x_1$. The neighbors of y_3 in C are y, x_2, x_4 and

(C, y_3) is a short 3-wheel. This proves that H_1 satisfies (iii) of Definition 9.9. □

A double star cutset S of G is *full* if S contains two adjacent nodes u, v and all their neighbors, i.e. $S = N(u) \cup N(v)$. One can check in polynomial time whether a graph contains a full double star cutset: simply check whether $G \setminus (N(u) \cup N(v))$ is connected for every pair of adjacent nodes u, v.

Lemma 9.13 *If an undominated bipartite graph G has a double star cutset, then it has a full double star cutset.*

Exercise 9.14 *Prove Lemma 9.13.*

PROCEDURE 3

Input: A bipartite graph F not containing a short 3-wheel.

Output: Either a 3-odd-path configuration of F, or a list of undominated induced subgraphs $F_1^*, \ldots, F_q^*$ of F where $q \leq |V^c(F)|^2|V^r(F)|^2$ with the following properties:

- The graphs $F_1^*, \ldots, F_q^*$ do not contain a double star cutset.
- If the input graph F is not balanced and contains a smallest unbalanced hole H that is clean in F, then one of the graphs in the list, say F_i^*, contains a smallest unbalanced hole H^* in $\mathcal{C}(H)$ that is clean in F_i^*.

Step 1 Delete dominated nodes in F until F becomes undominated. Let $\mathcal{M} = \{F\}$, $\mathcal{T} = \emptyset$.

Step 2 If $\mathcal{M}$ is empty, stop. Otherwise remove a graph R from $\mathcal{M}$. If R has no full double star cutset, add R to $\mathcal{T}$ and repeat Step 2. Otherwise, let $S = N(u) \cup N(v)$ be a full double star cutset of R. Let $R'_1, \ldots, R'_l$ be the connected components of $R \setminus S$ and let $R_1, \ldots, R_l$ be the corresponding blocks, that is R_i is the graph induced by $V(R'_i) \cup S$. Go to Step 3.

Step 3 Consider every pair of nonadjacent nodes u_p and v_q adjacent to u and v respectively. If there exist two distinct connected components of $R \setminus S$ that each contain neighbors of u_p and neighbors of v_q, there is a 3-odd-path configuration between u_p and v_q and F is not balanced. Otherwise go to Step 4.

Step 4 From each block R_i, remove dominated nodes until the resulting graph R_i^* becomes undominated. Add to $\mathcal{M}$ all the graphs R_i^* that contain at least one chordless path of length 3. Go to Step 2.

Lemma 9.15 *Let F be a bipartite graph that does not contain a short 3-wheel and let H be a smallest unbalanced hole that is clean in F.*

If Procedure 3, when applied to F, does not detect a 3-odd-path configuration in Step 3, then one of the graphs F_i^*, obtained as ouput of Procedure 3, contains an unbalanced hole H^* in $\mathcal{C}(H)$.

Proof: It suffices to show that, if H is clean in $R \in \mathcal{M}$, then one of the blocks R_i^* obtained from R contains an unbalanced hole H^* in $\mathcal{C}(H)$ that is clean in R_i^*.

Let $N(u) \cup N(v)$ be the full double star cutset of R, used to decompose R in Procedure 3. Let $R_1', \ldots, R_l'$ be the connected components of $R \setminus (N(u) \cup N(v))$ and $R_1, \ldots, R_l$ be the corresponding blocks. We first show that if no 3-odd-path configuration is detected in Step 3, an unbalanced hole $H^* \in \mathcal{C}(H)$ is contained in some block R_i.

Choose $H^* \in \mathcal{C}(H)$ such that $V(H^*) \cap \{u, v\}$ is maximal. By Lemma 9.12, H^* is clean in R, so u is either in H^* or has at most one neighbor in H^* and the same holds for v.

Let W be the subgraph induced by $V(H^*) - (N(u) \cup N(v))$. We have three possibilities for W:

(i) If H^* contains no neighbor of u and v, then $W = H^*$.

(ii) If both u and v have a single neighbor u_1 and v_1 in H^* and u_1, v_1 are nonadjacent, then W consists of two paths.

(iii) In all the remaining cases, it is easy to check that W consists of a single path.

If H^* does not belong to any of the blocks $R_1, \ldots, R_l$, the graph W must be disconnected and have a component in, say, R_i' and another in, say, R_j'. So (ii) holds. Let u_1 and v_1 be the neighbors of u and v in H^*. Then $V(W) \cup \{u, v\}$ induces a 3-odd-path configuration from u_1 to v_1 which is detected in Step 3 of the algorithm.

So, at the end of Step 3, one block R_i contains H^* and, by Lemma 9.12, $H^* \in \mathcal{C}(H)$ is clean in R_i. Since H^* is clean, the graph R_i^*, obtained from R_i by removing dominated nodes, contains a hole $H^{**} \in \mathcal{C}(H^*) = \mathcal{C}(H)$, where possibly $H^* = H^{**}$. □

Lemma 9.16 *The number of graphs $F_1^*, \ldots, F_q^*$ produced by Procedure 3 applied to F is bounded by $|V^r(F)|^2|V^c(F)|^2$. So is the number of double star cutsets used by Procedure 3.*

Proof: Let $N(u) \cup N(v)$ be a full double star cutset of $R \in \mathcal{M}$ used in Procedure 3. Let $R'_1, \ldots, R'_l$ be the connected components of $R \setminus (N(u) \cup N(v))$ and let $R^*_1, \ldots, R^*_l$ be the corresponding undominated blocks.

Claim 1 *No two distinct undominated blocks contain the same chordless path of length 3.*

Proof of Claim 1: Suppose by contradiction that a chordless path $P = a, b, c, d$ belongs to two distinct undominated blocks R^*_i and R^*_j. Then, in R, we have $\{a, b, c, d\} \subseteq N(u) \cup N(v)$.

Node u is distinct from a and d for otherwise a and d are adjacent and P is not a chordless path. By symmetry, v is also distinct from a and d. Since both R^*_i and R^*_j are undominated, both nodes a and d have at least one neighbor in both the connected components R'_i and R'_j. Now Step 3 of Procedure 3 detects a 3-odd-path configuration. This completes the proof of Claim 1.

Claim 2 *The graph R contains at least one chordless path of length 3 that is not contained in any of the undominated blocks R^*_i.*

Proof of Claim 2: Each of the connected components $R'_1, \ldots, R'_l$ must contain at least two nodes, since R is an undominated graph. At least one node in R'_i must be adjacent in R to a node in $N(u) \cup N(v)$. Assume w.l.o.g. that node p_i in R'_i is adjacent to a neighbor of v in R, say v_i.

Suppose now no node in R'_i is adjacent to a node in $N(u)$. Then the nodes in $N(u) - \{v\}$ are dominated by v. Thus, the undominated block R^*_i does not contain any neighbor of u except v. This in turn implies that node u is dominated by v_i. Thus u would have been deleted from R^*_i. Now $P = p_i, v_i, v, u$ is a chordless path of length 3 in R but P is not in any of the undominated blocks $R^*_1, \ldots, R^*_l$.

So a node in R'_i must be adjacent to a node that is a neighbor of u in R, say u_i. Repeating the same argument for $j = 1, \ldots, t$, it follows that each connected component R'_j contains a node, say w_j, that is adjacent in R to a node $u_j \in N(u)$. Suppose now u_j has a neighbor, say g in a connected component R'_k, distinct from R'_j. Let q be a neighbor of g in R'_k. Then $P = q, g, u_j, w_j$ is a chordless path of length 3 contained in R but not in any of the undominated blocks $R^*_1, \ldots, R^*_l$. Suppose now that u_j does not have any neighbor in R'_k, $k \neq j$. Then, in Step 4 of Procedure 3, node u_j is deleted from the undominated block R^*_k. Now the path w_k, u_k, u, u_j is a chordless path of length 3 contained in R but not in any of the undominated blocks $R^*_1, \ldots, R^*_l$. This completes the proof of Claim 2.

The lemma is obviously true if, after Step 1 has been applied, F does not have a full double star cutset. Now assume a full double star cutset exists. Every undominated block that is added to the list $\mathcal{M}$ in Step 4 of Procedure 3 contains a chordless path of length 3. Hence every undominated block that is added to the list $\mathcal{T}$ in Step 2 contains a chordless path of length 3. By Claim 1, the same chordless path of length 3 is not in any other undominated block that is added to the list $\mathcal{T}$. So the number of graphs in the list $F^*_1, \ldots, F^*_q$ is at most equal to the number of paths of length 3 in F, which is bounded by $|V^r(F)|^2|V^c(F)|^2$. By Claim 2, it follows that the number of full double star cutsets used to decompose the graph F with Procedure 3 is at most $|V^r(F)|^2|V^c(F)|^2$. □

Validity of the Algorithm

Theorem 9.17 *The running time of the recognition algorithm is polynomial in the size of the input graph G, and the algorithm correctly identifies G as balanced or not.*

Proof: The recognition algorithm described in Section 9.1.2 applies first Procedures 1, 2 and 3. The running time of each of these procedures has been shown to be polytime. Finally, in Step 4, the algorithm checks whether each of the (polynomially many) blocks is totally unimodular. Total unimodularity can be checked in polytime [173]. Hence the running time of the recognition algorithm described in Section 9.1.2 is polynomial.

Suppose G is balanced. Then G does not contain a short 3-wheel or a 3-odd-path configuration. All the induced subgraphs of G are balanced, so the graphs produced by Procedures 2 and 3 are balanced. Now, by Theorem 9.1, every graph in the list $B_1, \ldots, B_s$ is totally unimodular. Then Step 4 of the algorithm identifies G as balanced.

Suppose G is not balanced. If G contains a short 3-wheel, Step 1 of the algorithm identifies G as not balanced. Suppose G does not contain a short 3-wheel. Clearly G contains a smallest unbalanced hole. By Lemma 9.10, one of the induced subgraphs, say G_i, of G, in the list produced by Procedure 2 contains a smallest unbalanced hole H^* that is clean in G_i. Now G_i is one of the graphs considered for double star cutset decompositions by Procedure 3. By Lemma 9.15, Procedure 3 either detects a 3-odd-path configuration or one of the undominated blocks, say B_j, in the final list produced by Procedure 3 contains an unbalanced hole in the family $\mathcal{C}(H^*)$. In the former case G is correctly identified as not balanced. In the latter case, B_j is not totally unimodular and Step 4 of the algorithm identifies G as not balanced. □

9.2 Proof Outline of the Decomposition Theorem

9.2.1 Even Wheels

A wheel (H, v) is *even* if v has an even number (≥ 4) of neighbors in H. The proof of Theorem 9.1 involves two major cases depending on whether or not the graph contains an even wheel as an induced subgraph.

Theorem 9.18 *If a balanced bipartite graph G contains an even wheel as an induced subgraph, then G has a double star cutset.*

Given an even wheel (H, v), a subpath of H having two nodes of $N(v) \cap V(H)$ as endnodes and only nodes of $V(H) - N(v)$ as intermediate nodes is called a *sector* of (H, v). Two sectors are *adjacent* if they have a common endnode. We paint the nodes of $V(H)-N(v)$ with two colors, say blue and green, in such a way that nodes of $V(H)-N(v)$ have the same color if they are in the same sector, and have distinct colors if they are in adjacent sectors. The nodes of $N(v)\cap V(H)$ are left unpainted.

The proof of Theorem 9.18 involves two intermediate results about strongly adjacent nodes u to an even wheel (H,v). The first (Lemma 9.19) discusses the case when u and v are in opposite sides of the bipartition. The second (Lemma 9.20) discusses the case when u and v are on the same side.

Lemma 9.19 *Let (H,v) be an even wheel of a balanced bipartite graph G. Let u be a node in the opposite side of the bipartition as v such that u is not adjacent to v and u has neighbors in at least two distinct sectors of H. Then u has exactly two neighbors in H and they belong to sectors of the same color.*

Proof: The proof uses the fact that a balanced bipartite graph cannot contain an odd wheel nor a 3-odd-path configuration (recall Theorem 7.15). Assume that node u has neighbors in at least three different sectors, say S_i, S_j, S_k. If none of these sectors is adjacent to the other two, then there exist three unpainted nodes v_i, v_j, v_k, such that $v_i\in V(S_i)-(V(S_j)\cup V(S_k))$, $v_j\in V(S_j)-(V(S_i)\cup V(S_k))$, $v_k\in V(S_k)-(V(S_i)\cup V(S_j))$. This implies the existence of a 3-odd-path configuration from u to v where each of the nodes v_i, v_j, v_k belongs to a distinct path of the 3-odd-path configuration. If H has four sectors each containing neighbors of u, then each sector has exactly one neighbor of u (otherwise there is a 3-odd-path configuration from u to v). But now the graph induced by two adjacent sectors and u contains an odd wheel or a 3-odd-path configuration. So u has neighbors in exactly three sectors and one of them is adjacent to the other two, say S_j is adjacent to both S_i and S_k. Let v_i be the unpainted node in $V(S_i)\cap V(S_j)$ and v_k the unpainted node in $V(S_j)\cap V(S_k)$. Then, there is a 3-odd-path configuration from u to v unless node u has a unique neighbor u_i in S_i adjacent to v_i and a unique neighbor u_k in S_k adjacent to v_k, i.e. u,u_i,v_i,v,v_k,u_k,u is a 6-hole, a contradiction to balancedness.

So u has neighbors in at most two sectors of the wheel, say S_j and S_k. If these two sectors are adjacent, let v_i be their common endnode and v_j, v_k the other endnodes of S_j and S_k respectively. Let H' be the hole obtained from H by replacing $V(S_j) \cup V(S_k)$ by the shortest path in $V(S_j) \cup V(S_k) \cup \{u\} - \{v_i\}$. The wheel (H', v) is an odd wheel. So the sectors S_j and S_k are not adjacent.

If u has three neighbors or more on H, say two or more in S_j and at least one in S_k, then denote by v_j and v_{j-1} the endnodes of S_j and by v_k one of the endnodes of S_k. There exists a 3-odd-path configuration from u to v where each of the nodes v_j, v_{j-1}, and v_k belongs to a different path. Therefore u has only two neighbors in H, say $u_j \in V(S_j)$ and $u_k \in V(S_k)$. Let C_1 and C_2 be the holes formed by the node u and the two $u_j u_k$-subpaths of H, respectively. In order for both (C_1, v) and (C_2, v) to be even wheels, the sectors S_j and S_k must be of the same color. □

This proof gives a flavor of the arguments used in [51]. We state the next results without proofs, referring the reader to [51] for details.

An even wheel (H, v) is *small* if no even wheel of G contains strictly fewer nodes. Let $T(H, v)$ denote the set of nodes u such that at least two neighbors of u in H are adjacent to v and no sector of (H, v) entirely contains all the neighbors of u in H.

Lemma 9.20 *Let (H, v) be a small even wheel in a balanced bipartite graph. If u is in the same side of the bipartition as v and u is adjacent to both a blue and a green node, then u belongs to $T(H, v)$.*

The two above lemmas show that every node of G that is adjacent to a blue node and a green node in a small even wheel (H, v) belongs to $N(v) \cup T(H, v)$. This result has the following generalization.

Lemma 9.21 *Let (H, v) be a small even wheel in a balanced bipartite graph. Then every path connecting a blue node to a green node of (H, v) contains a node in $N(v) \cup T(H, v)$.*

Now Theorem 9.18 follows from the following result.

Lemma 9.22 *Let (H, v) be a small even wheel in a balanced bipartite graph. Then v has a neighbor y on H adjacent to all the nodes in $T(H, v)$.*

Indeed, Lemmas 9.21 and 9.22 imply that $N(v) \cup N(y)$ is a double star cutset in G. So Theorem 9.1 holds when G contains an even wheel. The most difficult part of the proof is to show that Theorem 9.1 also holds when G contains no even wheel. We refer the reader to [51].

9.2.2 A Conjecture

Double star cutset decompositions are not balancedness preserving and this heavily affects the running time of the algorithm for recognizing whether a $0, 1$ matrix is balanced (Section 9.1). It would be useful to strengthen Theorem 9.1 so that a balancedness preserving operation is used for the decomposition. Such a result is not known even for linear balanced bipartite graphs [58], and it may very well be that no such strengthening of Theorem 9.1 is possible. Theorem 9.1 can be refined by replacing double star cutsets by extended star cutsets, a concept introduced in [51], but this does not help in improving the complexity of the recognition algorithm. A further refinement may be possible, as follows.

A *biclique* is a complete bipartite graph where the two sides of the bipartition are both nonempty. The question arises whether Theorem 9.1 can be strengthened by showing that every balanced graph that is not totally unimodular has a biclique cutset.

The graph in Figure 9.2 shows that this is not always the case. More generally, define an infinite family of graphs as follows. Let H be a hole where nodes $u_1, \ldots, u_p, v_1, \ldots, v_q, w_1, \ldots, w_p, x_1, \ldots, x_q$ appear in this order when traversing H, but are not necessarily adjacent. Let $Y = \{y_1, \ldots, y_p\}$ and $Z = \{z_1, \ldots, z_q\}$ be two node sets having empty intersection with $V(H)$ and inducing a biclique K_{YZ}. Node y_i is adjacent to u_i and w_i for $1 \leq i \leq p$. Node z_i is adjacent to v_i and x_i for $1 \leq i \leq q$. Any balanced graph of this form for $p, q \geq 2$ is called a W_{pq}. For all values of $p, q \geq 2$, the graph W_{pq} is not totally unimodular and has no biclique cutset.

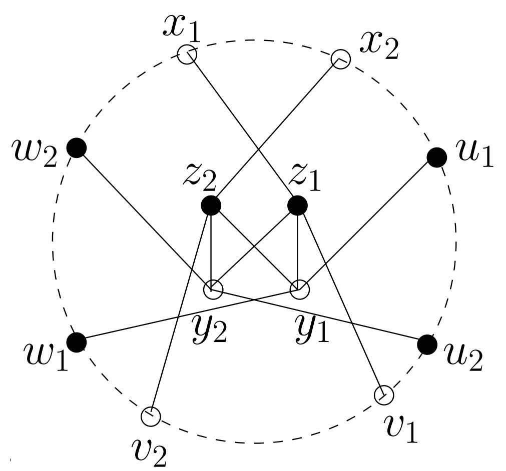

Figure 9.2: W_{22}

A graph G has a *2-join* if its node set can be partitioned into V_1, V_2 in such a way that, for each $i = 1, 2$, V_i contains disjoint nonempty node sets A_i and B_i such that every node of A_1 is adjacent to every node of A_2, every node of B_1 is adjacent to every node of B_2 and there are no other adjacencies between V_1 and V_2. Furthermore, for $i = 1, 2$, V_i contains at least one path from A_i to B_i, and if A_i and B_i are both of cardinality 1, then the graph induced by V_i is not a chordless path.

Conjecture 9.23 (Conforti, Cornuéjols, Rao [51]) *If G is a balanced graph that is not totally unimodular, then G is either a W_{pq} or has a biclique cutset or a 2-join.*

9.3 Balanced $0, \pm 1$ Matrices

In this section, we state a decomposition theorem for balanced $0, \pm 1$ matrices and outline a polynomial recognition algorithm [49].

Consider the following question: given a $0, 1$ matrix, is it possible to turn some of the 1's into -1's in order to obtain a balanced $0, \pm 1$ matrix? Camion's signing algorithm, which we studied in Chapter 7, gives the answer: this signing is essentially unique and easy to find. A

0, 1 matrix for which such a signing exists is called a *balanceable* matrix. So, in effect, the problem of recognizing whether a given $0, \pm 1$ matrix is balanced is equivalent to the problem of recognizing whether a 0, 1 matrix is balanceable.

9.3.1 Decomposition Theorem

2-Join and 6-Join

A 2-join was defined in Section 9.2.2. A bipartite graph G has a *6-join* if its node set can be partitioned into V_1, V_2 so that V_1 contains three disjoint nonempty node sets A_1, A_3, A_5, and V_2 contains three disjoint nonempty node sets A_2, A_4, A_6 such that, for $i = 1, \ldots, 6$, every node in A_i is adjacent to every node in $A_{i-1} \cup A_{i+1}$ (indices are taken modulo 6), and these are the only edges in the subgraph A induced by the node set $\cup_{i=1}^{6} A_i$. Furthermore, the only adjacencies between V_1 and V_2 are the edges of $E(A)$ and $|V_i| \geq 4$ for $i = 1, 2$.

Basic Classes of Graphs

A bipartite graph is *strongly balanceable* if it is balanceable and contains no cycle with exactly one chord. Strongly balanceable bipartite graphs can be recognized in polynomial time [56]. R_{10} is the balanceable bipartite graph defined by the cycle $x_1, \ldots, x_{10}, x_1$ of length 10 with chords $x_i x_{i+5}$, $1 \leq i \leq 5$.

Theorem 9.24 (Conforti, Cornuéjols, Kapoor and Vušković [49]) *A connected balanceable bipartite graph that is not strongly balanceable is either R_{10} or contains a 2-join, a 6-join or a double star cutset.*

The key idea in the proof of Theorem 9.24 is that if a balanceable bipartite graph G is not strongly balanceable, then an earlier result of Conforti, Cornuéjols and Rao [51] applies, or else G contains R_{10} or a connected 6-hole (to be defined next) and, in each case, Theorem 9.24 is shown to hold.

A *triad* consists of three internally node-disjoint paths $t, \ldots, u$; $t, \ldots, v$ and $t, \ldots, w$, where t, u, v, w are distinct nodes and u, v, w belong to the same side of the bipartition. Furthermore, the graph induced by the nodes of the triad contains no other edges than those of the three paths. Nodes u, v and w are called the *attachments*.

A *fan* consists of a chordless path $x, \ldots, y$ together with a node z adjacent to at least one node of the path, where x, y and z are distinct nodes all belonging to the same side of the bipartition. Nodes x, y and z are called the *attachments* of the fan.

A *connected 6-hole* Σ is a graph induced by two disjoint node sets $T(\Sigma)$ and $B(\Sigma)$ such that each induces either a triad or a fan, the attachments of $B(\Sigma)$ and $T(\Sigma)$ induce a 6-hole and there are no other adjacencies between the nodes of $T(\Sigma)$ and $B(\Sigma)$.

Theorem 9.25 [49] *A connected balanceable bipartite graph that contains R_{10} as a proper induced subgraph has a biclique cutset.*

Theorem 9.26 [49] *A balanceable bipartite graph that contains a connected 6-hole has a double star cutset or a 6-join.*

Theorem 9.27 [51] *A balanceable bipartite graph not containing R_{10} or a connected 6-hole as induced subgraphs either is strongly balanceable or contains a 2-join or a double star cutset.*

Now Theorem 9.24 follows from Theorems 9.25, 9.26 and 9.27.

Theorem 9.24 may have the following strengthening.

Conjecture 9.28 [49] *A connected balanceable bipartite graph that is not W_{pq}, R_{10} or strongly balanceable has a 2-join, a 6-join or a biclique cutset.*

Another direction in which Theorem 9.24 might be strengthened is as follows.

Conjecture 9.29 [49] *A balanceable bipartite graph that is not regular has a double star cutset.*

9.3.2 Recognition Algorithm

To a $0, \pm 1$ matrix, we associate its signed bipartite representation obtained by assigning a weight $+1$ or -1 to its edges in the natural way. We will outline an algorithm for checking whether a signed bipartite graph is balanced. By Theorem 9.24, a connected balanced signed bipartite graph is strongly balanced (no cycle has a unique chord) or is a signed copy of R_{10} or it contains a 2-join, a 6-join or a double star cutset.

Consider a connected signed bipartite graph G where S is a node cutset or an edge cutset. We construct signed bipartite graphs, called blocks, from the connected components of $G \setminus S$ and we say that the resulting decomposition is *balancedness preserving* if all the blocks are balanced if and only if G itself is balanced. The idea of the algorithm is to decompose G using balancedness preserving decompositions into a polynomial number of basic blocks that can each be checked for balancedness in polynomial time.

For the 2-join and 6-join, the blocks can be defined so that the decompositions are balancedness preserving. For the double star cutset this is not immediately possible.

2-Join Decomposition

Let $K_{A_1A_2}$ and $K_{B_1B_2}$ define a 2-join of G such that neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique. Let V_1, V_2 be the corresponding partition of $V(G)$. We construct two blocks G_1 and G_2 from G as follows. For $i = 1, 2$, let P_i be a shortest path from A_i to B_i in $G(V_i)$, . Define H_1 to be the graph induced by $V_1 \cup V(P_2)$. Similarly H_2 is the graph induced by $V_2 \cup V(P_1)$. For $i = 1, 2$, construct G_i from H_i as follows. Replace P_i by a path M_i with same endnodes, with intermediate nodes of degree two, with length $4 \leq |E(M_i)| \leq 5$ and edge weights $+1$ or -1 chosen so that the weight of M_i is congruent to the weight of P_i modulo 4.

Theorem 9.30 *Let G_1 and G_2 be the blocks of the decomposition of the signed bipartite graph G by a 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$. If neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique and G does not contain an unbalanced hole of length 4, then G is balanced if and only if both G_1 and G_2 are balanced.*

6-Join Decomposition

Let G be a signed bipartite graph that has a 6-join $E(A)$. *Blocks* G_1 and G_2 of a 6-join decomposition are constructed as follows. For $i = 1, \ldots, 6$ let a_i be any node of A_i. G_1 is a subgraph of G induced by the node set $V_1 \cup \{a_2, a_4, a_6\}$ and G_2 is a subgraph of G induced by the node set $V_2 \cup \{a_1, a_3, a_5\}$.

Theorem 9.31 *Let G_1 and G_2 be the blocks of the decomposition of the signed bipartite graph G by a 6-join $E(A)$. If G does not contain an unbalanced hole of length 4 or 6, then G is balanced if and only if both G_1 and G_2 are balanced.*

Double Star Cutset Decomposition

As in the case of balanced 0,1 matrices, a "cleaning step" is used. See [49] for details.

Theorem 9.32 *In a clean signed bipartite graph containing no short 3-wheel, at least one block of a double star decomposition is clean.*

This theorem guarantees that, after the cleaning step has been performed, double star decompositions do not break a smallest unbalanced hole.

Algorithm

The recognition algorithm takes a signed bipartite graph as input and recognizes whether or not the graph is balanced. The algorithm consists of four phases:

- **Preprocessing** The input graph is first checked for a short 3-wheel. If it contains one, it is not balanced. Otherwise, a polynomial number of signed subgraphs are generated, at least one of which is clean. All of these subgraphs are balanced if and only if the input graph is balanced.

- **Double Stars** The second phase consists in doing double star decompositions, until no block contains a double star cutset.

- **6-joins** The third phase consists in doing 6-join decompositions until no block contains a 6-join.
- **2-joins** The last phase consists in doing 2-join decompositions until no block contains a 2-join.

The 2-join and 6-join decompositions do not create any new double star cutset in the blocks except in one case that can be dealt with easily. Also a 2-join decomposition does not create any new 6-join in any of the blocks. As a result when the algorithm terminates none of the blocks created have a double star cutset, 2-join or 6-join decomposition. By the decomposition theorem (Theorem 9.24), if the original signed bipartite graph was balanced the blocks must be strongly balanced or signed copies of R_{10}. R_{10} is a graph with only ten nodes and so can be checked in constant time. Checking whether a signed bipartite graph is strongly balanced can be done in polynomial time (Conforti and Rao [56]). The preprocessing phase and the phases of decomposition using 2-joins and 6-joins can be shown to be polynomial. For the double star decomposition it is shown that the graph has a path with three edges that is not present in any of the blocks. This bounds the number of such decompositions by a polynomial in the size of the graph. Thus the entire algorithm is polynomial time bounded.

Chapter 10

Decomposition of Perfect Graphs

Much research has been devoted to studying classes of perfect graphs that can be decomposed into "basic classes" using "perfection-preserving operations". In this chapter, we present four basic classes of perfect graphs and several operations on graphs that preserve perfection. In the last section, we illustrate the decomposition approach on a class of perfect graphs called Meyniel graphs.

10.1 Basic Classes

Bipartite graphs are perfect since, for any induced subgraph H, if $\omega(H) = 2$, then $\chi(H) = 2$ and if $\omega(H) = 1$, then $\chi(H) = 1$.

A graph L is the *line graph* of a graph G if $V(L) = E(G)$ and two nodes of L are adjacent if and only if the corresponding edges of G are adjacent.

Theorem 10.1 *Line graphs of bipartite graphs are perfect.*

Proof: If L is the line graph of graph G, then $\chi(L) = \chi'(G)$, where χ denotes the chromatic number and χ' the edge-chromatic number. Let ω denote the clique number and Δ the largest degree. Then $\omega(L) = \Delta(G)$, except in the trivial case where $\Delta(G) = 2$ and G contains a triangle.

If G is bipartite, $\chi'(G) = \Delta(G)$ by Gupta's theorem [116] and therefore $\chi(L) = \omega(L)$. Since induced subgraphs of L are also line graphs of bipartite graphs, the result follows. $\square$

Since bipartite graphs and line graphs of bipartite graphs are perfect, it follows from Lovász's perfect graph theorem (Theorem 3.4) that complements of bipartite graphs and of line graphs of bipartite graphs are perfect. To summarize, in this section we have introduced four basic classes of perfect graphs:

- bipartite graphs and their complements, and
- line graphs of bipartite graphs and their complements.

10.2 Perfection-Preserving Compositions

A *composition* of two graphs G_1 and G_2 is an operation that constructs a third graph G where each of G_1 and G_2 has fewer nodes than G. We write $G = G_1 * G_2$. Conversely, a graph G can be $*$-*decomposed* if there exist graphs G_1 and G_2 such that $G = G_1 * G_2$. Composition $*$ is said to *preserve perfection* if G_1 and G_2 are perfect if and only if $G_1 * G_2$ is perfect.

The following compositions preserve perfection.

Union Given graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \emptyset$, define G to have node set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

Clique identification (Berge [8]) Let K_1 and K_2 be cliques of the same cardinality in graphs G_1 and G_2 respectively, where $|K_i| < |V(G_i)|$ for $i = 1, 2$. Label the nodes so that $V(G_1) \cap V(G_2) = K_1 = K_2$ and define G to have node set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Union is the special case of clique identification where $K_1 = K_2 = \emptyset$. To see that clique identification preserves perfection, note that $\omega(G) = \max(\omega(G_1), \omega(G_2))$ and that $\omega(G)$-colorings of G_1 and G_2 can be composed into an $\omega(G)$-coloring of G. Finding whether a graph has a clique articulation can be done in $O(n^3)$, where n is the number of nodes in the graph, as shown by Whitesides [209].

Join (Cunningham and Edmonds [68]) Let G_1 and G_2 be graphs, each with at least three nodes, such that $V(G_1) \cap V(G_2) = \emptyset$, and let $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. Denote by $N(v_i)$ the set of neighbors of v_i in G_i, for $i = 1, 2$. The *join* of G_1 and G_2 is the graph G with node set $(V(G_1) - \{v_1\}) \cup (V(G_2) - \{v_2\})$ obtained from $G_1 \setminus \{v_1\}$ and $G_2 \setminus \{v_2\}$ by connecting all nodes in $N(v_1)$ to all nodes in $N(v_2)$. The fact that the join preserves perfection was proved by Bixby [14]. The proof goes as follows. Replicate node v_i in G_i a number of times equal to $\omega(G) - p_i$ where p_i is the size of a largest clique in $N(v_i)$, for $i = 1, 2$. Since these graphs are perfect by the replication lemma (Lemma 3.3), they can be $\omega(G)$-colored. Furthermore, only p_i distinct colors appear on the nodes of $N(v_i)$. This implies that these two ω-colorings can be composed into an ω-coloring of G. Cunningham [67] gave an $O(n^3)$ algorithm to find whether a graph has a join decomposition.

Amalgam (Burlet and Fonlupt [21]) Let K_1 and K_2 be cliques of the same cardinality (possibly empty) in graphs G_1 and G_2 respectively. Let $v_1 \in V(G_1) - K_1$, $v_2 \in V(G_2) - K_2$ be such that all nodes of K_i are adjacent to all nodes of $\{v_i\} \cup N(v_i)$, for $i = 1, 2$, where $N(v_i)$ denotes the set of neighbors of v_i in G_i. Label the nodes so that $V(G_1) \cap V(G_2) = K_1 = K_2$. Finally, assume that $V(G_i) - K_i$ has cardinality at least three for $i = 1, 2$. The *amalgam* of G_1 and G_2 is the graph G with node set $(V(G_1) - \{v_1\}) \cup (V(G_2) - \{v_2\})$ obtained from $G_1 \setminus \{v_1\}$ and $G_2 \setminus \{v_2\}$ by adding edges connecting all the nodes in $N(v_1)$ to all nodes in $N(v_2)$. Note that the join is a special case of the amalgam. Burlet and Fonlupt [21] showed that the amalgam composition preserves perfection using a proof similar to that for the join. Cornuéjols and Cunningham [60] gave an $O(mn^2)$ algorithm to find whether a graph has an amalgam decomposition, where n is the number of nodes and m is the number of edges.

Exercise 10.2 *Show that the amalgam composition preserves perfection.*

2-Amalgam (Cornuéjols and Cunningham [60]) A graph G has a *2-amalgam* if its node set can be partitioned into V_1, V_2 and K in such a way that, for each $i = 1, 2$, V_i contains disjoint nonempty node sets A_i and B_i such that every node of A_1 is adjacent to every node of A_2, every node of B_1 is adjacent to every node of B_2, there are no other adjacencies between V_1 and V_2, the nodes of K induce a clique and are adjacent to all the nodes in $A_1 \cup A_2 \cup B_1 \cup B_2$ and possibly other nodes of $V_1 \cup V_2$. Also, for $i = 1, 2$, V_i contains at least one path from A_i to B_i, and if A_i and B_i are both of cardinality 1, then the graph induced by V_i is not a chordless path. When $K = \emptyset$, the 2-amalgam is called a *2-join*.

We construct two blocks G_1 and G_2 from G as follows. For $i = 1, 2$, let P_i be a shortest path from A_i to B_i in $G(V_i)$. Define G_1 to be the graph induced by $V_1 \cup K \cup V(P_2)$. Similarly G_2 is the graph induced by $V_2 \cup K \cup V(P_1)$. Conversely, given G_1 and G_2, the 2-amalgam composition is the operation that constructs G such that G_1, G_2 are the blocks of a 2-amalgam of G. There is an $O(m^2n^2)$ algorithm to find whether a graph has a 2-amalgam decomposition, where n is the number of nodes and m is the number of edges. Next we show that the 2-amalgam preserves perfection ([60]; see also Kapoor [129] Chapter 8).

Theorem 10.3 *Let G be a graph with a 2-amalgam and let G_1 and G_2 be the blocks of a 2-amalgam decomposition. G is perfect if and only if G_1 and G_2 are perfect.*

Proof: By definition, G_1 and G_2 are induced subgraphs of G. It follows that, if G is perfect, so are G_1 and G_2. Now we prove the converse

$$(*) \qquad \text{If } G_1 \text{ and } G_2 \text{ are perfect, then so is } G.$$

Claim 1: It suffices to prove $(*)$ in the case where $K = \emptyset$.

Proof of Claim 1: Assume $K \neq \emptyset$. Suppose $(*)$ does not hold, i.e, G_1 and G_2 are perfect but G is not. Let H be a minimally imperfect subgraph of G. Clearly H is not a subgraph of G_1 or G_2, so $V(H)$ has a nonempty intersection with both V_1 and V_2. Suppose H contains a node $x \in K$. Since H does not have a star cutset by Theorem 3.34, it follows that $V(H)$ is contained in $K \cup A_1 \cup B_1$ or $K \cup A_2 \cup B_2$. But then the complement of H is a disconnected graph, a contradiction. So H contains no node of K. Let H_1 and H_2 be subgraphs of G_1 and G_2 induced by $V(G_1) \cap V(H)$ and $V(G_2) \cap V(H)$ respectively. Graphs H_1 and H_2 are perfect and H is the 2-join of H_1 and H_2. Indeed, since H is distinct from H_1 and H_2, is connected and has no join, it verifies all the assumptions required in the definition of a 2-join. This proves Claim 1.

In the rest of the proof, we assume that $K = \emptyset$. The proof of $(*)$ is based on a coloring argument, combining colorings of the perfect graphs G_1 and G_2 (Claim 3) into a coloring of G (Claim 4). To prove Claim 3, we will use the following result.

Claim 2: Let uv be an edge in G_i such that $N(u) \cap N(v) = \emptyset$. Let G'_i be the graph obtained from G_i by duplicating node v into v'. Let H_i be the graph obtained from G'_i by deleting edge u, v'. G_i is perfect if and only if H_i is perfect.

Proof of Claim 2: Graph G_i is an induced subgraph of H_i. It follows that, if H_i is perfect, then so is G_i.

Conversely, suppose G_i is perfect and H_i is not. Let H^* be a minimally imperfect subgraph of H_i. Let G^* be the subgraph of G'_i induced by the nodes in H^*. Since G^* is perfect but H^* is not, $V(H^*)$ must contain nodes u and v'. Also H^* and G^* have the same chromatic number but the size of a maximum clique in G^* is one greater than a maximum clique in H^*. Therefore the maximum clique in G^* is uvv'. No node of $N(v) - \{u\}$ is in H^* since otherwise H^* would also have a clique of cardinality three. Now $\{v\}$ is a clique cutset of H^* separating v' from the rest of the graph, a contradiction to the assumption that H^* is minimally imperfect. This proves Claim 2.

Let ω_i be the size of a maximum clique in G_i. Let a_i and b_i be the sizes of a maximum clique in A_i and B_i respectively. In a coloring of G_i, let $C(A_i)$ and $C(B_i)$ be the sets of colors used in the coloring of the nodes in A_i and B_i respectively.

Claim 3: If G_i is a perfect graph and $\omega \geq \omega_i$, then there exists a coloring of $G(V_i)$ with at most ω colors such that $|C(A_i)| = a_i$, $|C(B_i)| = b_i$ and

(i) if P_i has an odd number of edges, then $|C(A_i) \cap C(B_i)| = \max(0, a_i + b_i - \omega)$,

(ii) if P_i has an even number of edges, then $|C(A_i) \cap C(B_i)| = \min(a_i, b_i)$,

Proof of Claim 3: We consider the two cases separately.

(i) P_i has an odd number of edges.

Let $P_i = x_1, \ldots, x_{2k}$. Duplicate node x_{2k} into x'_{2k} and remove edge $x_{2k-1}x'_{2k}$. By Claim 2, the new graph is perfect. For i odd, $1 \leq i < 2k$, duplicate node x_i $\omega - a_i$ times. For i even, $1 < i \leq 2k-2$, duplicate node x_i a_i times.

If $a_i + b_i < \omega$, duplicate x_{2k} a_i times and duplicate x'_{2k} $\omega - a_i - b_i$ times. The size of a maximum clique in the new graph, say H, is ω and H is perfect. Color H using ω colors. Note that every node of P_i belongs to a clique of size ω. So, in the coloring, the colors that appear in the duplicates of x_{2k-1} do not appear in $C(A_i)$. But then the colors that appear in the duplicates of x_{2k} are precisely $C(A_i)$. Therefore the nodes in B_i and the duplicates of x'_{2k} are colored using colors that do not appear in $C(A_i)$. Thus $|C(A_i) \cap C(B_i)| = 0$.

If $a_i + b_i \geq \omega$, duplicate x_{2k} $\omega - b_i$ times and remove x'_{2k}. The size of a maximum clique in the new graph H is ω and H is perfect. Color H using ω colors. Again, in the coloring, the colors that appear in the duplicates of x_{2k-1} do not appear in $C(A_i)$. These colors cannot appear in the duplicates of x_{2k}, so they must appear in $C(B_i)$. So the number of common colors in $C(A_i)$ and $C(B_i)$ is $b_i - (\omega - a_i) = a_i + b_i - \omega$.

(ii) P_i has an even number of edges.

Assume w.l.o.g. that $a_i \leq b_i$. Let $P_i = x_1, \ldots, x_{2k+1}$. For i odd, $1 \leq i \leq 2k-1$, duplicate node x_i $\omega - a_i$ times. For i even, $1 < i \leq 2k$, duplicate node x_i a_i times. Finally, duplicate x_{2k+1} $\omega - b_i$ times. The new graph H is perfect and the size of a maximum clique in H is ω. Color H using ω colors. The colors that appear in $C(A_i)$ are precisely the colors that appear in the duplicates of x_{2k}. But then these colors do not appear in the duplicates of x_{2k+1} and consequently must appear in $C(B_i)$. Thus $|C(A_i) \cap C(B_i)| = \min(a_i, b_i)$. This proves Claim 3.

Claim 4: If G_1 and G_2 are perfect, then G is perfect.

Proof of Claim 4: Since G_1 contains no odd hole, every chordless path from A_1 to B_1 has the same parity as P_1. It follows from the definition of 2-amalgam decomposition that P_1 and P_2 have the same parity.

Let ω be the size of a maximum clique in G. We will construct a coloring of G with ω colors. This will be sufficient to prove Claim 4 since, if Claim 4 fails, it also fails for a minimally imperfect graph G (recall the argument in Claim 1). Clearly, $\omega \geq a_1 + a_2$ and $\omega \geq b_1 + b_2$.

(i) P_1 and P_2 both have an odd number of edges.

Then by Claim 3 (i), there exists a coloring of G_i with $|C(A_i) \cap C(B_i)| = \max(0, a_i + b_i - \omega)$. In the coloring of G_1, label by 1 through a_1 the colors that occur in A_1 and by ω through $\omega - b_1 + 1$ the colors that occur in B_1. In the coloring of G_2, label by ω through $\omega - a_2 + 1$ the colors that occur in A_2 and by 1 through b_2 the colors that occur in B_2. Color $G(V_1)$ and $G(V_2)$ to conform to the colorings of G_1 and G_2. If this is not a valid coloring of G, there must exist a common color in A_1 and A_2 or in B_1 and B_2. But then either $a_1 \geq \omega - a_2 + 1$ or $b_2 \geq \omega - b_1 + 1$, a contradiction.

(ii) P_1 and P_2 both have an even number of edges.

Then by Claim 3 (ii), there exists a coloring of G_i with $|C(A_i) \cap C(B_i)| = \min(a_i, b_i)$. In the coloring of G_1, label by 1 through a_1 the colors that occur in A_1 and by 1 through b_1 the colors that occur in B_1. In the coloring of G_2, label by ω through $\omega - a_2 + 1$ the colors that occur in A_2 and by ω through $\omega - b_2 + 1$ the colors that occur in B_2. Color $G(V_1)$ and $G(V_2)$ to conform to the colorings of G_1 and G_2. If this is not a valid coloring of G, there must exist a common color in A_1 and A_2 or in B_1 and B_2. But then either $a_1 \geq \omega - a_2 + 1$ or $b_1 \geq \omega - b_2 + 1$, a contradiction. □

In addition to perfection-preserving decompositions, some decompositions that do not preserve perfection seem to be important in the study of perfect graphs. This should not be surprising since a similar situation occured in Chapter 9 for balanced matrices. For example, Chvátal, Fonlupt, Sun and Zemirline [33] use the following decomposition.

Rosette A *z-edge* is any edge whose endnodes are both adjacent to a node z. A graph G has a *rosette* centered at node z if the graph obtained from $G \setminus z$ by removing all z-edges is disconnected and the subgraph of G induced by all the neighbors of z consists of node disjoint cliques. Note that, if G is not a clique, a rosette is a special case of a star cutset, another important cutset in the theory of perfect graphs (see Theorem 3.34).

Exercise 10.4 *Let A be the clique-node matrix of a graph G and let $B(A)$ be the bipartite representation of the 0,1 matrix A. Give the precice connection between*

(i) a clique cutset in G and a star cutset in $B(A)$,

(ii) a rosette in G and a star cutset in $B(A)$,

(iii) a 2-amalgam with $K \neq \emptyset$ in G and a biclique cutset in $B(A)$.

Describe the decomposition of G that corresponds to a double star cutset in $B(A)$.

10.3 Meyniel Graphs

Theorem 10.5 (Meyniel [144]) *If, in a graph, every odd cycle of length five or greater has at least two chords, then the graph is perfect.*

A graph is called a *Meyniel graph* if every odd cycle of length five or greater has at least two chords. Clearly, the definition is not a good characterization of Meyniel graphs since we need to enumerate all the odd cycles to know whether or not a graph is a Meyniel graph. Burlet and Fonlupt [21] give a polynomial algorithm to recognize these graphs. It is based on a decomposition of Meyniel graphs into basic graphs using the amalgam decomposition recursively.

Exercise 10.6 *Let G be the amalgam of G_1 and G_2. Show that, if G_1 and G_2 are Meyniel graphs, then G is a Meyniel graph.*

A graph is *triangulated* if it contains no hole. The structure of triangulated graphs is well studied, see e.g. [107] and there are efficient recognition algorithms to test membership in this class. Clearly, triangulated graphs are Meyniel graphs.

A node u is *universal* for a subgraph K of $G \setminus u$ if u is adjacent to every node of K. Clearly, bipartite graphs are Meyniel graphs. So are bipartite graphs B plus a universal node for B.

A *basic Meyniel graph* G is either a triangulated graph or a bipartite graph B together with at most one universal node for B.

Theorem 10.7 (Burlet and Fonlupt [21]) *If a connected Meyniel graph is not basic, then it can be amalgam decomposed.*

This theorem yields a polynomial algorithm to recognize Meyniel graphs: Starting from G, we recursively repeat the amalgam decomposition on each of the graphs obtained in the process until no further amalgam decomposition exists (the resulting graphs are called the *blocks* of the decomposition). Finding an amalgam decomposition in a graph can be done in polynomial time [60]. Conforti-Gerards [53] show that the number of blocks is linear. Finally, there is a polynomial algorithm for recognizing whether the blocks are basic Meyniel graphs.

Exercise 10.8 *Show that basic Meyniel graphs can be decomposed into the basic classes introduced in Section 10.1 using clique cutsets and the complement of the union operation. [Hint: Use the fact that a triangulated graph is either a clique or contains a clique cutset.]*

Theorem 10.7 was generalized to cap-free graphs [50]. A *cap* is a hole together with a node adjacent to exactly two consecutive nodes in the hole. Clearly, Meyniel graphs are cap-free, since a cap contains an odd hole or is an odd cycle with a unique chord.

Exercise 10.9 *Show that a cap-free graph is perfect if and only if it is a Meyniel graph.*

A *basic cap-free graph* G is either a triangulated graph or a biconnected triangle-free graph B together with at most one universal node for B.

In this section we prove the following:

Theorem 10.10 (Conforti, Cornuéjols, Kapoor, Vušković [50]) *If a connected cap-free graph is not basic, then it can be amalgam decomposed.*

We follow the proof in [50].

10.3.1 D-structures

Let $G(S)$ denote the subgaph of G induced by the subset S of V.

Definition 10.11 *A* D-structure (C_1, C_2, K) *of* G *consists of disjoint sets of nodes* C_1, C_2 *and* K*, where* $|C_1| \geq 2$*,* $|C_2| \geq 2$ *and the nodes of* K *induce a clique of* G *(possibly* K *is empty). Furthermore, the subgraph* $G(C_1)$ *is connected and every node in* C_1 *is universal for* $C_2 \cup K$*, every node in* C_2 *is universal for* $C_1 \cup K$ *and there exists no node in* $V - (C_1 \cup C_2 \cup K)$ *adjacent to both a node in* C_1 *and a node in* C_2*.*

Lemma 10.12 *If a cap-free graph* G *contains a D-structure then* G *contains an amalgam decomposition.*

To prove this lemma we first need to prove the following result:

Lemma 10.13 *Let* $G(V')$ *be a connected subgraph of a cap-free graph* G*,* $|V'| \geq 2$*. Let* z *be a node universal for* V' *and let* y *be a node that is adjacent but not universal to* V' *such that* y *and* z *are connected by some chordless path* P' *in* $G(V - V')$*. Then there exists a node* $x \in V(P')$*, adjacent but not universal to* V' *such that, in the subpath* P *of* P' *from* z *to* x*, all the nodes in* $V(P) - \{x\}$ *are universal for* V'*.*

Proof: In P' pick x to be the node closest to z that is adjacent but not universal to V'. Let P be the xz-subpath of P'. Let x' be the node closest to x in P universal for V'. If x' is not adjacent to x, then the subpath of P connecting x to x', together with two adjacent nodes $u, v \in V'$, where v is adjacent to x and u is not adjacent to x forms a cap. Note that since V' is connected, such a choice of nodes u and v is always possible. Now let x'' be the node of P not adjacent to V' closest to x. Pick the subpath P'' of P containing x'' with only the endnodes universal for V'. Let w be the node of $V(P) - V(P'')$ adjacent to the endnode of P'' closest to x. Now P'' together with w and a node of V' adjacent to w induces a cap. □

Proof: (Lemma 10.12) Let U be the set of nodes in $V - (C_1 \cup C_2 \cup K)$ that are adjacent to C_1 and are connected to a node in C_2 by a path in $G(V - (C_1 \cup K))$.

Claim 1: *Every node in U is universal for C_1.*

Proof: Assume not and let $u \in U$ be connected to $y \in C_2$ by a chordless path P_u in $G(V - (C_1 \cup K))$. Since C_1 and C_2 belong to a D-structure, then the length of P_u is greater than one. By Lemma 10.13, we may assume that all the nodes of P_u, except for u, are universal for C_1. Now the node on P_u adjacent to y has neighbors in both C_1 and C_2, contradicting the definition of a D-structure. This completes the proof of Claim 1.

Let K' contain the nodes in K that are not universal for U and $K'' = K - K'$. Define $A = C_1$, $B = C_2 \cup K' \cup U$. We show that (A, B, K'') is an amalgam decomposition of G. Claim 1 shows that every node in B is universal for A and by definition of K'', every node in K'' is universal for U. Since (C_1, C_2, K) is a D-structure, every node in K'' is universal for $C_1 \cup C_2 \cup K'$.

Claim 2: *Let G' be the graph obtained from G by removing all edges with one endnode in A and the other in K'. Then in $G'(V - (C_2 \cup K'' \cup U))$ no path connects a node of K' and a node of $C_1 = A$.*

Proof: Let $P = k, \ldots, v_k, x$ be a chordless path connecting $k \in K'$ and $x \in C_1$ and contradicting the claim. No intermediate node of P is adjacent to a node in C_2 else, by Claim 1, v_k belongs to U, contradicting the definition of P. If P has length greater than 2, then the nodes of P, together with any node in C_2 induce a cap.

So $P = k, v_k, x$. Since k is not universal for U, there exists a node $u \in U$ not adjacent to k. Let $P_u = x_1, \ldots, x_m$ be a chordless path connecting $u = x_1$ and a node $x_m \in C_2$ in $G(V - (C_1 \cup K))$. Let $u = u_1, \ldots, u_n = x_m$ be the nodes of P_u that are universal for C_1 with u_i closer to u than u_{i+1}. Note that all nodes $u_1, \ldots, u_{n-1}$ belong to U.

We now show that u_i cannot be adjacent to u_{i+1}, $1 \leq i \leq n-1$. Assume not and let i be the highest index such that u_i and u_{i+1} are adjacent. If $i = n-1$, u_i contradicts the definition of a D-structure. So $i < n-1$. Then the nodes in the subpath of P_u between u_{i+1} and u_{i+2}, together with u_i and any node in C_1 induce a cap.

Let x_j be the node of smallest index adjacent to k. (Since x_m is adjacent to k, such a node exists). Since u is not adjacent to k, $j > 1$. If x_j is universal for C_1, let x_i be the node of U having largest index $i < j$. Now the nodes in the subpath of P_u between x_i and x_j, together with k and any node of C_1 induce a cap. So x_j is not universal for C_1.

Let x_i be the node of $V(P_u) \cap U$ having largest index $i < j$. Now the nodes in the subpath of P_u between x_i and x_j, together with k, v_k and x induce a cap. (Note that node v_k is not adjacent to any node in P_u since otherwise v_k belongs to U, contradicting the assumption). This completes the proof of Claim 2.

The following claim shows that (A, B, K'') is an amalgam decomposition of G.

Claim 3: *Let G'' be obtained from G by removing all edges with one endnode in A and the other in B. Then in $G''(V - K'')$, no path connects a node in A and a node in B.*

Proof: Let $P = x_1, \ldots, x_n$ be a chordless path between x_1 in A and x_n in B and contradicting the claim. Claim 1 shows that if $x_n \in C_2$, then $x_2 \in U$, a contradiction. Claim 2 shows $x_n \notin K'$. So $x_n \in U$ and let P_{x_n} be a path connecting x_n and a node in C_2 in $G(V - (C_1 \cup K))$. Now there is a path in $G(V - (C_1 \cup K))$ between x_2 and a node in C_2 only using nodes of $V(P_{x_n}) \cup V(P)$. So x_2 must belong to U, a contradiction. □

10.3.2 M-structures

M-structures were first introduced by Burlet and Fonlupt [21] in their study of Meyniel graphs.

An induced subgraph $G(V_1)$ of G is called an *M-structure* (multipartite structure) if $\bar{G}(V_1)$ contains at least two connected components each with at least two nodes. Let $W_1, \ldots, W_k$ be the node sets of these connected components. The *proper subclasses* of $G(V_1)$ are the sets W_i of cardinality greater than or equal to 2. The *partition* of an M-structure is denoted by $(W_1, \ldots, W_r, K)$ where K is the union of all non-proper subclasses. Note that K induces a clique in G.

Lemma 10.14 *An M-structure $G(V_1)$ of G is maximal with respect to node inclusion, if and only if there exists no node $v \in V - V_1$ such that v is universal for a proper subclass of $G(V_1)$.*

Proof: Let $G(V_1 \cup \{u\})$ be an M-structure. Assume node u is not universal for any proper subclass of $G(V_1)$. In $\bar{G}(V_1 \cup \{u\})$ node u is adjacent to at least one node in each of the proper subclasses. Thus there exists only one proper subclass in $G(V_1 \cup \{u\})$, contradicting the assumption.

Conversely let node u be universal for some proper subclass W_i of $G(V_1)$. Then $\bar{G}(V_1 \cup \{u\})$ has at least two components with more than one node, the graph induced by W_i and at least one component with more than one node in $(V_1 \cup \{u\}) - W_i$. $\square$

The above proof yields the following:

Corollary 10.15 *Let $G(V_1)$ and $G(V_2)$ be M-structures with $V_1 \subseteq V_2$. Let W_i and Z_j be connected components of $\bar{G}(V_1)$ and $\bar{G}(V_2)$ respectively having nonempty intersection. Then $W_i \subseteq Z_j$.*

Lemma 10.16 *Let $G(V_1)$ be a maximal M-structure of a cap-free graph G. Then a node in $V - V_1$ cannot be adjacent to two proper subclasses of $G(V_1)$.*

Proof: Assume node $u \in V - V_1$ is adjacent to two proper subclasses W_1 and W_2 of $G(V_1)$. Since $G(V_1)$ is maximal, by Lemma 10.14 node u is not universal for either of the classes. Also since the complement of $G(W_1)$ is connected, there must exist a pair of nodes x_1, y_1 adjacent in the complement, such that node u is adjacent to x_1 but not to y_1. Similarly there must exist a pair x_2, y_2 in W_2 such that x_2, y_2 are adjacent in the complement and node u is adjacent to x_2 but not to y_2. But now x_1, x_2, y_1, y_2 together with node u induce a cap. $\square$

Theorem 10.17 *If G is a cap-free graph containing an M-structure either with at least three proper subclasses, or with at least one proper subclass which is not a stable set, then G contains an amalgam decomposition.*

Proof: If G contains a D-structure (C_1, C_2, K) then, by Lemma 10.12, G contains an amalgam decomposition. So the theorem follows from the proof of the following statement:

If G is a cap-free graph containing an M-structure either with at least three proper subclasses, or with at least one proper subclass which is not a stable set, then G contains a D-structure (C_1, C_2, K).

Let $G(V_1)$ be an M-structure of G satisfying the above property and $G(V_2)$ a maximal M-structure with $V_1 \subseteq V_2$.

Claim 1: *The M-structure $G(V_2)$ either contains at least three proper subclasses or contains exactly two proper subclasses not both of which are stable sets.*

Proof: If $G(V_1)$ contains a proper subclass, say W_i, which is not a stable set, by Corollary 10.15, there exists a proper subclass, say Z_j of $G(V_2)$ such that $W_i \subseteq Z_j$. Then Z_j is not a stable set. If all proper subclasses of $G(V_1)$ are stable sets, then $G(V_1)$ has at least three proper subclasses say $W_1, W_2, \ldots, W_k$. If $G(V_2)$ has only two proper subclasses, say Z_1, Z_2, then by Corollary 10.15, we may assume w.l.o.g. that $W_1 \cup W_2 \subseteq Z_1$. Then Z_1 is not a stable set, since every node in W_1 is adjacent to a node in W_2. This completes the proof of Claim 1.

Claim 2: *Let $G(V_2)$ be a maximal M-structure of G with partition (W_1, W_2, K), where W_1 is not a stable set. Then G contains a D-structure (C_1, C_2, K).*

Proof: Let C_1 be a connected component of $G(W_1)$ with more than one node. Let $C_2 = W_2$. Then (C_1, C_2, K) is a D-structure, since by Lemma 10.16 no node of $V - V_2$ is adjacent to a node in C_1 and a node in C_2, and $|C_2| \geq 2$, since W_2 is a proper subclass of $G(V_2)$. This completes the proof of Claim 2.

Claim 3: *Let $G(V_2)$ be a maximal M-structure of G with at least three proper subclasses. Then G contains a D-structure (C_1, C_2, K).*

Proof: Let $W_1, W_2, \ldots, W_l$, $l \geq 3$ be the proper subclass of $G(V_2)$ and let K be the collection of all non-proper subclasses. Let C_1 be the nodes in two proper subclasses of $G(V_2)$, (note that $G(C_1)$ is a connected graph), C_2 be the nodes in all the other proper subclasses of $G(V_2)$. Then (C_1, C_2, K) is a D-structure since $|C_1| \geq 2$, $|C_2| \geq 2$ and Lemma 10.16 shows that the only nodes having neighbors in both C_1 and C_2 belong to K. So the proof of Claim 3 is complete. $\square$

10.3.3 Expanded Holes

An *expanded hole* consists of disjoint nonempty sets of nodes $S_1, \ldots, S_n$, $n \geq 4$, not all singletons, such that, for all $1 \leq i \leq n$, the graphs $G(S_i)$

are connected. Furthermore, every $s_i \in S_i$ is adjacent to $s_j \in S_j$, $i \neq j$, if and only if $j = i+1$ or $j = i-1$ (modulo n).

Lemma 10.18 *Let G be a cap-free graph and let H be a hole of G. If s is a node having two adjacent neighbors in H, then either s is universal for H or s together with H induces an expanded hole.*

Proof: Let s be a node with two adjacent neighbors in H. If s has no other neighbors on H, then s induces a cap with H. Let $H = x_1, \ldots, x_n, x_1$ with node s adjacent to x_1 and x_n. If s is not universal for H, and does not induce an expanded hole together with H, then let k be the smallest index for which s is not adjacent to x_k. Let l be the smallest index such that $l > k$ and s is adjacent to x_l. Now node x_{k-2} (x_n if $k = 2$) together with the hole $s, x_{k-1}, \ldots, x_l, s$ forms a cap. □

Lemma 10.19 *Let G be a cap-free graph and let $S = \cup_{i=1}^{n} S_i$, $n > 4$, be a maximal expanded hole in G with respect to node inclusion. Either G contains an M-structure with a proper subclass which is not a stable set of G, or all nodes that are adjacent to a node in S_i and a node in S_{i+1} ($S_{n+1} = S_1$) for some i are universal for S and induce a clique of G.*

Proof: Let u be a node adjacent to $s_1 \in S_1$ and $s_2 \in S_2$. By applying Lemma 10.18 to any hole that contains s_1 and s_2 and a node each from the sets S_j, $j > 2$, we have that u is adjacent to all nodes in $S - (S_1 \cup S_2)$, else the maximality of S is contradicted. Now since node u is adjacent to s_1, s_2 and is universal for all sets S_j, $j > 2$, Lemma 10.18 shows that u is universal for S_1 and S_2, hence for S.

Let u and v be two nonadjacent nodes that are universal for S. Then u, v together with $s_1 \in S_1$, $s_2 \in S_2$ and $s_4 \in S_4$ induces an M-structure with proper sets $W_1 = \{u, v\}$ and $W_2 = \{s_1, s_2, s_4\}$. Furthermore W_2 is not a stable set of G. □

Theorem 10.20 *A cap-free graph that contains an expanded hole contains an amalgam decomposition.*

Proof: Let $S = \cup_{i=1}^{n} S_i$ be a maximal expanded hole in G. First assume that $n = 4$. Then the node set S induces an M-structure with proper subclasses $S_1 \cup S_3$ and $S_2 \cup S_4$. $S_2 \cup S_4$ is not a stable set because, say, $|S_2| \geq 2$ and $G(S_2)$ is connected. Hence by Theorem 10.17 we are done.

Now assume that $n > 4$. By Lemma 10.12, it is sufficient to show that G contains a D-structure (C_1, C_2, K). Assume w.l.o.g. that $|S_2| \geq 2$ and let K be the set of nodes that are universal for S. Lemma 10.19 shows that K is a clique of G. Let $C_1 = S_2$ and $C_2 = S_1 \cup S_3$. Lemma 10.19 shows that every node that is adjacent to a node of C_1 and a node of C_2 is universal for S and hence belongs to K. Therefore (C_1, C_2, K) is a D-structure. □

10.3.4 The Main Theorem

Now we are ready to prove Theorem 10.10, which we restate here for convenience.

Theorem *Every connected cap-free graph that does not contain an amalgam decomposition is basic cap-free.*

Proof: Assume G does not contain an amalgam decomposition and is not a basic cap-free graph. Since G is not triangulated, G contains a nonempty biconnected triangle-free subgraph. Let F be a maximal node set inducing such a biconnected triangle-free subgraph.

Claim 1: *Every node in $V - F$ that has at least two neighbors in F is universal for F.*

Proof: Let u be a node in $V - F$ having at least two neighbors in F. The graph induced by $F \cup \{u\}$ contains a triangle u, x, y else the maximality of F is contradicted. Let H be a hole in $G(F)$ containing x and y. (H exists since, by biconnectedness, x and y belong to a cycle and since $G(F)$ contains no triangle, the smallest cycle containing x and y is a hole). Lemma 10.18 shows that either u is universal for H or forms an expanded hole with H. Theorem 10.20 rules out the latter possibility. Let $F' \subseteq F$ be a maximal set of nodes such that $G(F')$ contains H, is biconnected and such that node u is universal for F'. If $F \neq F'$, then since $G(F)$ and $G(F')$ are biconnected, some $z \in F - F'$ belongs to a hole that contains an edge of $G(F')$. Let H' be such a hole. By Lemma 10.18 and Theorem 10.20, node u is adjacent to all the nodes of H'. Let $F'' = F' \cup V(H')$. $G(F'')$ is biconnected, u is universal for F''. Hence F'' contradicts the maximality of F'. Hence u is universal for F and the proof of Claim 1 is complete.

Claim 2: *Let U be the set of universal nodes for F. Then the nodes in U induce a clique of G.*

Proof: Let $w, z \in U$ be two nonadjacent nodes of U and let $v_1, \ldots, v_n, v_1$ be a hole of $G(F)$. Then nodes w, z together with v_1, v_2, v_3 and v_4 induce an M-structure, either with two proper subclasses not both of which are stable if v_1 and v_4 are not adjacent, or with three proper subclasses. By Theorem 10.17, G contains an amalgam decomposition. This completes the proof of Claim 2.

Claim 3: $V = F \cup U$.

Proof: Let $S = V - (F \cup U)$. By Claim 1, every node in S has at most one neighbor in F. Let C be a connected component of $G(S)$. By maximality of F, there is at most one node in F, say y, that has a neighbor in C. If such a node y exists, let $C_1, \ldots, C_l$ be the connected components of $G(S)$ adjacent to y. Let $V_1 = C_1 \cup \ldots C_l \cup \{y\}$, $A = \{y\}$, $K = U$, $V_2 = V - (V_1 \cup K)$ and B be the set of neighbors of y in F. Then (A, B, K) is an amalgam decomposition of G, separating V_1 from V_2.

If no component of $G(S)$ is adjacent to a node of F, let $V_1 = U \cup S$, $A = U$, $V_2 = B = F$. Then $(A, B, \emptyset)$ is an amalgam decomposition of G. This completes the proof of Claim 3.

If U contains at least two nodes, then let $V_1 = A = U$, $V_2 = B = F$ and $(A, B, \emptyset)$ is an amalgam decomposition of G. If U contains at most one node, then G is a basic cap-free graph. $\square$

Bibliography

[1] R. Anstee and M. Farber, Characterizations of Totally Balanced Matrices, *Journal of Algorithms 5* (1984) 215-230.

[2] K. Appel and W. Haken, Every Planar Map is Four Colorable. Part I: Discharging, *Illinois Journal of Mathematics 21* (1977) 429-490.

[3] F. Barahona, The Max Cut Problem in Graphs Not Contractible to K_5, *Operations Research Letters 2* (1983) 107-111.

[4] C. Berge, Les Problèmes de Coloration en Théorie des Graphes, *Publication de l'Institut de Statistique de l'Université de Paris 9* (1960) 123-160.

[5] C. Berge, Färbung von Graphen deren sämtliche bzw. deren ungerade Kreise starr sind (Zusammenfassung), Wissenschaftliche Zeitschrift, Martin Luther Universität Halle-Wittenberg, Mathematisch-Naturwissenschaftliche Reihe (1961) 114-115.

[6] C. Berge, Sur Certains Hypergraphes Généralisant les Graphes Bipartis, in *Combinatorial Theory and its Applications I* (P. Erdös, A. Rényi and V. Sós eds.), *Colloq. Math. Soc. János Bolyai 4*, North Holland, Amsterdam (1970) 119-133.

[7] C. Berge, Balanced Matrices, *Mathematical Programming 2* (1972) 19-31.

[8] C. Berge, *Graphs and Hypergraphs*, North Holland (1973).

[9] C. Berge, Balanced Matrices and the Property G, *Mathematical Programming Study 12* (1980) 163-175.

[10] C. Berge, Minimax Theorems for Normal and Balanced Hypergraphs. A Survey, in: *Topics on perfect graphs* (C. Berge and V. Chvátal eds.), *Annals of Discrete Mathematics 21* (1984) 3-21.

[11] C. Berge, Minimax Relations for the Partial q-Colorings of a Graph, *Discrete Mathematics 74* (1989) 3-14.

[12] C. Berge and P. Duchet, Strongly Perfect Graphs, in: *Topics on Perfect Graphs* (C. Berge and V. Chvátal eds.), *Annals of Discrete Mathematics 21*, North Holland, Amsterdam (1984) 221-224.

[13] C. Berge and M. Las Vergnas, Sur un Théorème du Type König pour les Hypergraphes, *Annals of the New York Academy of Sciences 175* (1970) 32-40.

[14] R.E. Bixby, A Composition for Perfect Graphs, in: *Topics on Perfect Graphs* (C. Berge and V. Chvátal eds.), *Annals of Discrete Mathematics 21*, North Holland, Amsterdam (1984) 221-224.

[15] R.E. Bixby and W.H. Cunningham, Converting Linear Programs to Network Problems, *Mathematics of Operations Research 5* (1980) 321-357.

[16] R.E. Bixby and A. Rajan, A Short Proof of the Truemper-Tseng Theorem on Max-Flow Min-Cut Matroids, *Linear Algebra and its Applications 114* (1989) 277-292.

[17] R.G. Bland, H.C. Huang and L.E. Trotter, Jr., Graphical Properties Related to Minimal Imperfection, *Discrete Math. 27* (1979) 11-22.

[18] E.O. Boros and O. Čepek, On Perfect $0, \pm 1$ Matrices, *Discrete Mathematics 165* (1997) 81-100.

[19] W.G. Bridges and H.J. Ryser, Combinatorial Designs and Related Systems, *J. Algebra 13* (1969) 432-446.

[20] T.H. Brylawski, A Decomposition for Combinatorial Geometries, *Trans. Amer. Math. Soc. 171* (1972), 235-282.

[21] M. Burlet and J. Fonlupt, Polynomial Algorithm to Recognize a Meyniel Graph, in: *Topics on Perfect Graphs* (C. Berge and V. Chvátal eds.), *Annals of Discrete Mathematics 21*, North Holland, Amsterdam (1984) 225-252.

[22] M. Burlet and J.-P. Uhry, Parity Graphs, *Annals of Discrete Mathematics 16* (1982) 1-26.

[23] M. Cai, X. Deng and W. Zang, A TDI System and its Application to Approximation Algorithm, *Proc. 39th IEEE Symposium on Foundations of Computer Science* (1998).

[24] M. Cai, X. Deng and W. Zang, A Min-Max Theorem on Feedback Vertex Sets, *Integer Programming and Combinatorial Optimization*, 7th IPCO Conference, Graz (1999).

[25] K. Cameron and J. Edmonds, Existentially Polynomial Theorems, *DIMACS Series in Dicrete Mathematics and Theoretical Computer Science 1*, American Mathematical Society, Providence, R.I (1990) 83-100.

[26] P. Camion, Caractérisation des Matrices Unimodulaires, *Cahier du Centre d'Etudes de Recherche Opérationelle 5* (1963) 181-190.

[27] P. Camion, Characterization of Totally Unimodular Matrices, *Proceedings of the American Mathematical Society 16* (1965) 1068-1073.

[28] V. Chandru, J.N. Hooker, Extended Horn sets in Propositional Logic, *Journal of the ACM 38* (1991) 205-221.

[29] V. Chandru and J.N. Hooker, *Optimization Methods for Logical Inference*, Wiley, New York (1999).

[30] V. Chvátal, On Certain Polytopes Associated with Graphs, *Journal of Combinatorial Theory B 18* (1975) 138-154.

[31] V. Chvátal, Perfectly Orderable Graphs, in: *Topics on Perfect Graphs* (C. Berge and V. Chvátal eds.), *Annals of Discrete Mathematics 21*, North Holland, Amsterdam (1984) 63-65.

[32] V. Chvátal, Star-Cutsets and Perfect Graphs, *J. Combin. Theory Ser B.* 39 (1985) 189-199.

[33] V. Chvátal, J. Fonlupt, L. Sun and A. Zemirline, Recognizing Dart-Free Perfect Graphs, *Symposium on Discrete Algorithms* (2000).

[34] V. Chvátal, R. L. Graham, A.F. Perold and S.H. Whitesides, Combinatorial Designs Related to the strong Perfect Graph Conjecture, *Discrete Math.* 26 (1979) 83-92.

[35] V. Chvátal, N. Sbihi, Bull-free Berge Graphs are Perfect, *Graphs and Combinatorics* 3 (1987) 127-139.

[36] P Codato, M. Conforti, C. Serafini, Packing T-Joins, *Journal of Graph Theory 22* (1996) 293-296.

[37] F. G. Commoner, A Sufficient Condition for a Matrix to be Totally Unimodular, *Networks 3* (1973) 351 365.

[38] M. Conforti, private communication (1996).

[39] M. Conforti, private communication (2000).

[40] M. Conforti and G. Cornuéjols, A Decomposition Theorem for Balanced Matrices, *Integer Programming and Combinatorial Optimization*, R. Kannan and W.R. Pulleyblank eds., Waterloo University Press (1990) 147-169.

[41] M. Conforti and G. Cornuéjols, Clutters that Pack and the Max Flow Min Cut Property: A Conjecture, *The Fourth Bellairs Workshop on Combinatorial Optimization*, W.R. Pulleyblank and F.B. Shepherd eds. (1993).

[42] M. Conforti and G. Cornuéjols, Balanced $0, \pm 1$ Matrices, Bicoloring and Total Dual Integrality, *Mathematical Programming 71* (1995) 249-258.

[43] M. Conforti and G. Cornuéjols, A Class of Logic Problems Solvable by Linear Programming, *Journal of the ACM 42* (1995) 1107-1113.

[44] M. Conforti, G. Cornuéjols and C. de Francesco, Perfect $0, \pm 1$ Matrices, *Linear Algebra and its Applications 43* (1997) 299-309.

[45] M. Conforti, G. Cornuéjols, G. Gasparyan and K. Vušković, Perfect Graphs, Partitionable Graphs and Cutsets, preprint (1999).

[46] M. Conforti, G. Cornuéjols, A. Kapoor, M.R. Rao and K. Vušković, Balanced Matrices, *Math. Programming, State of the Art 1994* (J. R. Birge, K. G Murty, eds.) (1994) 1-33.

[47] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, Perfect Matchings in Balanced Hypergraphs, *Combinatorica 16* (1996) 325-329.

[48] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, Perfect, Ideal and Balanced Matrices, *Annotated Bibliographies in Combinatorial Optimization*, M. Dell'Amico, F. Maffioli, S. Martello eds., Wiley (1997) 81-94.

[49] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, Balanced $0, \pm 1$ Matrices, Part I: Decomposition, and Part II: Recognition Algorithm, preprints, Carnegie Mellon University (revised 1998).

[50] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković, Even and Odd Holes in Cap-Free Graphs, *Journal of Graph Theory 30* (1999) 289-308.

[51] M. Conforti, G. Cornuéjols and M.R. Rao, Decomposition of Balanced Matrices, *Journal of Combinatorial Theory B 77* (1999) 292-406.

[52] M. Conforti, G. Cornuéjols, K. Truemper, From Totally Unimodular to Balanced $0, \pm 1$ Matrices: A Family of Integer Polytopes, *Mathematics of Operations Research 19* (1994) 21-23.

[53] M. Conforti and A.M.H. Gerards, Stable Sets and Graphs with No Even Holes, preprint (1999).

[54] M. Conforti, A.M.H. Gerards and A. Kapoor, A Theorem of Truemper, to appear in *Combinatorica*, preliminary version by M.

Conforti and A. Kapoor, in *Lecture Notes in Computer Science: Integer Programming and Combinatorial Optimization*, 6th IPCO Conference, Houston (1998), 53-68.

[55] M. Conforti and E.L. Johnson, Two Min-Max Theorems for Graphs Noncontractible to a Four Wheel, preprint, IBM Thomas J. Watson Research Center (1987).

[56] M. Conforti and M. R. Rao, Structural Properties and Recognition of Restricted and Strongly Unimodular Matrices, *Mathematical Programming 38* (1987) 17-27.

[57] M. Conforti and M. R. Rao, Structural Properties and Decomposition of Linear Balanced Matrices, *Mathematical Programming 55* (1992) 129-168.

[58] M. Conforti, M. R. Rao, Properties of Balanced and Perfect Matrices, *Mathematical Programming 55* (1992) 35-47.

[59] M. Conforti and M. R. Rao, Testing Balancedness and Perfection of Linear Matrices, *Mathematical Programming 61* (1993) 1-18.

[60] G. Cornuéjols and W. H. Cunningham, Compositions for Perfect Graphs, *Discrete Mathematics 55* (1985) 245-254.

[61] G. Cornuéjols and B. Guenin, Ideal Binary Clutters, Connectivity and a Conjecture of Seymour, preprint (2000).

[62] G. Cornuéjols and B. Guenin, Ideal Clutters, preprint (2000), to appear in the special Survey Collection of *Discrete Applied Mathematics* devoted to DO'99.

[63] G. Cornuéjols and B. Guenin, A Note on Dijoins, preprint (2000).

[64] G. Cornuéjols, B. Guenin and F. Margot, The Packing Property, in *Integer Programming and Combinatorial Optimization*, R.E. Bixby, E. A. Boyd and R. Z. Rios-Mercado eds., *Lecture Notes in Computer Science 1412* (1998) 1-8. To appear in *Mathematical Programming*.

[65] G. Cornuéjols and B. Novick, Ideal 0,1 Matrices, *Journal of Combinatorial Theory B 60* (1994) 145-157.

[66] G. Cornuéjols and L.F. Zuluaga, On Padberg's Conjecture about Almost Totally Unimodular Matrices, to appear in *Operations Research Letters*.

[67] W.H. Cunningham, Decomposition of Directed Graphs, *SIAM journal on Algebraic and Discrete Methods 3* (1982) 214-228.

[68] W. H. Cunningham and J. Edmonds, A Combinatorial Decomposition Theory, *Canadian Journal of Mathematics 22* (1980) 734-765.

[69] E. Dahlhaus, J. Kratochvil, P.D. Manuel and M. Miller, Transversal Partitioning in Balanced Hypergraphs, *Discrete Applied Mathematics 79* (1997) 75-89.

[70] G.B. Dantzig, *Linear Programming and Extensions*, Princeton University Press (1963).

[71] N.G. de Bruijn and P. Erdös, On a Combinatorial Problem, *Proc. Kon. Ned. Akad. v. Wetensch 51* 1277-1279.

[72] M. Dell'Amico, F. Maffioli, S. Martello, *Annotated Bibliographies in Combinatorial Optimization*, Wiley (1997).

[73] R. Diestel, *Graph Theory*, *Graduate Texts in Mathematics 173*, Springer (1997).

[74] R.P. Dilworth, A Decomposition Theorem for Partially Ordered Sets, *Annals of Mathematics 51* (1950) 161-166.

[75] G. Ding, personal communication (1989).

[76] G. Ding, Clutters with $\tau_2 = 2\tau$, *Discrete Mathematics 115* (1993) 141-152.

[77] G. Ding and W. Zang, Packing Cycles in Graphs, preprint (1999).

[78] R.J. Duffin, The Extremal Length of a Network, *Journal of Mathematical Analysis and Applications 5* (1962) 200-215.

[79] J. Edmonds, Edge-Disjoint Branchings, in: *Combinatorial Algorithms* (B. Rustin, ed.), Academic Press, New York (1973) 91-96.

[80] J. Edmonds and D.R. Fulkerson, Bottleneck Extrema, *Journal of Combinatorial Theory B 8* (1970) 299-306.

[81] J. Edmonds and R. Giles, A Min-Max Relation for Submodular Functions on Graphs, *Annals of Discrete Mathematics 1* (1977) 185-204.

[82] J. Edmonds and E.L. Johnson, Matchings, Euler Tours and the Chinese Postman Problem, *Mathematical Programming 5* (1973) 88-124.

[83] J. Fonlupt and A. Zemirline, A Polynomial Recognition Algorithm of Perfect $K_4 \setminus \{e\}$ Graphs, preprint, IMAG-ARTEMIS, University of Grenoble (1987).

[84] L.R. Ford, Jr and D.R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton (1962).

[85] A. Frank, Edge-Disjoint Paths in Planar Graphs, *Journal of Combinatorial Theory B 39* (1985) 164-178.

[86] A. Frank, A. Sebö and E. Tardos, Covering Directed and Odd Cuts, *Mathematical Programming Study 22* (1984) 99-112.

[87] A. Frank and Z. Szigeti, On Packing T-Cuts, *Journal of Combinatorial Theory B 61* (1994) 363-371.

[88] D.R. Fulkerson, Blocking Polyhedra, in: *Graph Theory and its Applications* (B. Harris ed.), Academic Press, New York (1970) 93-112.

[89] D.R. Fulkerson, Blocking and Antiblocking Pairs of Polyhedra, *Mathematical Programming 1* (1971) 168-194.

[90] D. R. Fulkerson, Anti-Blocking Polyhedra, *Journal of Combinatorial Theory B 12* (1972) 50-71.

[91] D.R. Fulkerson, Packing Rooted Directed Cuts in a Weighted Directed Graph, *Mathematical Programming 6* (1974) 1-13.

[92] D. R. Fulkerson, A. Hoffman and R. Oppenheim, On Balanced Matrices, *Mathematical Programming Study 1* (1974) 120-132.

[93] T. Gallai, Transitiv Orientierbare Graphen, *Acta Mathematicae Academiae Scientarum Hungaricae 18* (1967) 25-66.

[94] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman (1979).

[95] G.S. Gasparyan, Minimal Imperfect Graphs: A Simple Approach, *Combinatorica 16* (1996) 209-212.

[96] G.S. Gasparyan, M. Preissmann and A. Sebö, Imperfect and Nonideal Clutters: A Common Approach, preprint (1999).

[97] J.F. Geelen, A Generalization of Tutte's Characterization of Totally Unimodular Matrices, *Journal of Combinatorial Theory B 70* (1997) 101-117.

[98] G. Georgakopoulos, D. Kavvadias, C. H. Papadimitriou, Probabilistic Satisfiability, *Journal of Complexity 4* (1988) 1-11.

[99] A.M.H. Gerards, A Short Proof of Tutte's Characterization of Totally Unimodular Matrices, *Linear Algebra and its Applications 114/115* (1989) 207-212.

[100] A.M.H. Gerards, *Graphs and Polyhedra, Binary Spaces and Cutting Planes*, CWI Tract 73, Amsterdam (1990).

[101] A.M.H. Gerards, An Orientation Theorem for Graphs, *Journal of Combinatorial Theory B 62* (1994) 199-212.

[102] A.M.H. Gerards, On Tutte's Characterization of Graphic Matroids – a Graphic Proof, *Journal of Graph Theory 20* (1995) 351-359.

[103] A.M.H. Gerards and A. Schrijver, Matrices with the Edmonds-Johnson Property, *Combinatorica 6* (1986) 365-379.

[104] A. Ghouila-Houri, Charactérisations des Matrices Totalement Unimodulaires, *C. R. Acad. Sc. Paris 254* (1962) 1192-1193.

[105] M.X. Goemans and V.S. Ramakrishnan, Minimizing Submodular Functions over Families of Sets, *Combinatorica 15* (1995) 499-513.

[106] M. C. Golumbic and C. F. Goss, Perfect Elimination and Chordal Bipartite Graphs, *Journal of Graph Theory 2* (1978) 155-163.

[107] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York (1980).

[108] M. Grötschel and W.R.Pulleyblank, Weakly Bipartite Graphs and the Max-Cut Problem, *Operations Research Letters 1* (1981) 23-27.

[109] B. Guenin, Perfect and Ideal $0, \pm 1$ Matrices, *Mathematics of Operations Research 23* (1998) 322-338.

[110] B. Guenin, A Characterization of Weakly Bipartite Graphs, in *Integer Programming and Combinatorial Optimization*, R.E. Bixby, E. A. Boyd and R. Z. Rios-Mercado eds., *Lecture Notes in Computer Science 1412* (1998) 9-22.

[111] B. Guenin, On Packing and Covering Polyhedra, PhD dissertation, Carnegie Mellon University (1998).

[112] B. Guenin, Integral Polyhedra Related to Even Cycle and Even Cut Matroids, preprint, Fields Institute, Toronto (2000).

[113] B. Guenin and R. Thomas, Packing Directed Circuits Exactly, *Mathematical Programming Symposium* (2000), Atlanta.

[114] M. Grötschel, L. Lovász and A. Schrijver, Polynomial Algorithms for Perfect Graphs,, in: *Topics on Perfect Graphs* (C. Berge and V. Chvátal eds.), *Annals of Discrete Mathematics 21*, North Holland, Amsterdam (1984) 325-356.

[115] M. Grötschel, L. Lovász and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, Heidelberg (1988).

[116] R. P. Gupta, An Edge-Coloration Theorem for Bipartite Graphs of Paths in Trees, *Discrete Mathematics 23* (1978) 229-233.

[117] P. Hall, On Representatives of Subsets, *J. London Math. Soc.*, 26-30.

[118] D.B. Hartvigsen and D.K. Wagner, Recognizing Max-Flow Min-Cut Path Matrices, *Operations Research Letters 7* (1988) 37-42.

[119] R.B. Hayward, Weakly Triangulated Graphs, *Journal of Combinatorial Theory B 39* (1985) 200-209.

[120] I. Heller, C.B. Tompkins, An Extension of a Theorem of Dantzig's, in: *Linear Inequalities and Related Systems* (H.W. Kuhn and A.W. Tucker eds.), Princeton University Press, Princeton, N.J. (1956) 247-254.

[121] A.J. Hoffman, A. Kolen and M. Sakarovitch, Characterizations of Totally Balanced and Greedy Matrices, *SIAM journal of Algebraic and Discrete Methods 6* (1985) 721-730.

[122] A.J Hoffman and J.B. Kruskal, Integral Boundary Points of Convex Polyhedra, in: *Linear Inequalities and Related Systems* (H.W. Kuhn and A.W. Tucker eds.), Princeton University Press, Princeton, N.J. (1956) 223-246.

[123] I. Holyer, The NP-Completeness of Edge Coloring, *SIAM Journal on Computing 10* (1981) 718-720.

[124] J.N. Hooker, A Quantitative Approach to Logical Inference, *Decision Support Systems 4* (1988) 45-69.

[125] J.N. Hooker, Resolution and the Integrality of Satisfiability Polytopes, *Mathematical Programming 74* (1996) 1-10.

[126] T.C. Hu, Multicommodity Network Flows, *Operations Research 11* (1963) 344-360.

[127] Y. Ikebe and A. Tamura, Perfect Bidirected Graphs, Report CSIM-2 Department of Computer Science and Information Mathematics, University of Electro-Communications, Tokyo (1996).

[128] E.L. Johnson and M. Padberg, Degree-Two Inequalities, Clique Facets and Biperfect Graphs, *Annals of Discrete Mathematics 16* (1985) 169-187.

[129] A. Kapoor, *On the Structure of Balanced Matrices and Perfect Graphs*, PhD dissertation, Carnegie Mellon University (1994).

[130] D. König, Graphs and Matrices, *Mat. Fiz. Lapok 38* (1931) 116-119 (in Hungarian).

[131] A. Lehman, A Solution to the Shannon Switching Game, *J. SIAM 12* (1964) 687-725.

[132] A. Lehman, On the Width-Length Inequality, mimeographic notes (1965), published: *Mathematical Programming 17* (1979) 403-417.

[133] A. Lehman, On the Width-Length Inequality and Degenerate Projective Planes, unpublished manuscript (1981), published: *Polyhedral Combinatorics* (W. Cook and P.D. Seymour eds.), *DIMACS Series in Discrete Mathematics and Theoretical Computer Science 1*, American Mathematical Society, Providence, R.I. (1990) 101-105.

[134] W.J. Li, Degree Two Inequalities and Biperfect Graphs, PhD Thesis, State University of New York, Stony Brook (1995).

[135] M.V. Lomonosov, Combinatorial Approaches to Multiflow Problems, *Discrete Applied Mathematics 11* (1985) 1-94.

[136] L. Lovász, Normal Hypergraphs and the Perfect Graph Conjecture, *Discrete Mathematics 2* (1972) 253-267.

[137] L. Lovász, A Characterization of Perfect Graphs, *Journal of Combinatorial Theory B 13* (1972) 95-98.

[138] L. Lovász, On Two Minimax Theorems in Graph Theory, *Journal of Combinatorial Theory B 21* (1976) 96-103.

[139] L. Lovász, On the Shannon Capacity of a Graph, *IEEE Transactions on Information Theory IT-25* (1979) 1-7.

[140] L. Lovász and M. D. Plummer, *Matching Theory*, *Annals of Discrete Mathematics 29*, North Holland, Amsterdam (1986).

[141] C.L. Lucchesi and D.H. Younger, A Minimax Relation for Directed Graphs, *Journal of the London Mathematical Society (2) 17* (1978) 369-374.

[142] C. Lütolf and F. Margot, A Catalog of Minimally Nonideal Matrices, *Mathematical Methods of Operations Research 47* (1998) 221-241.

[143] S.E. Markossian and I.A. Karapetian, On Critically Imperfect Graphs, in *Prikladnaia Matematika*, Erevan University (1984), (in Russian).

[144] H. Meyniel, On the Perfect Graph Conjecture, *Discrete Mathematics 16* (1976) 339-342.

[145] E.F. Moore and C.E. Shannon, Reliable Circuits Using Less Reliable Relays, *Journal of the Franklin Institute 262* (1956) 204-205.

[146] G. L. Nemhauser and L. A. Wolsey, *Integer and Combinatorial Optimization*, Wiley (1988).

[147] N. J. Nilsson, Probabilistic logic, *Artificial Intelligence 28* (1986) 71-87.

[148] P. Nobili and A. Sassano, $(0, \pm 1)$ Ideal Matrices, *Mathematical Programming 80* (1998) 265-281.

[149] B. Novick, Ideal 0,1 Matrices, Ph.D. dissertation, Carnegie Mellon University (1990).

[150] B. Novick and A. Sebö, On Combinatorial Properties of Binary Spaces, *Integer Programming and Combinatorial Optimization*, E. Balas and J. Clausen eds., *Lecture Notes in Computer Science 920* (1995) 212-227.

[151] H. Okamura and P.D. Seymour, Multicommodity Flows in Planar Graphs, *Journal of Combinatorial Theory B 31* (1981) 75-81.

[152] G.I. Orlova and Y.G. Dorfman, Finding the Maximum Cut in a Graph, *Izvestija Akademii Nauk SSSR, Tehničeskaja Kibernetika 3* (1972) 155-159 (in Russian). English translation: *Engineering Cybernetics 10* (1972) 502-506.

[153] J.G. Oxley, *Matroid Theory*, Oxford University Press (1992).

[154] M.W. Padberg, Perfect Zero-One Matrices, *Mathematical Programming 6* (1974) 180-196.

[155] M.W. Padberg, Almost Integral Polyhedra Related to Certain Combinatorial Optimization Problems, *Linear Algebra and Its Applications 15* (1976) 69-88.

[156] M.W. Padberg, Total Unimodularity and the Euler-Subgraph Problem, *Operations Research Letters 7* (1988) 173-179.

[157] M.W. Padberg, Lehman's Forbidden Minor Characterization of Ideal 0,1 Matrices, *Discrete Mathematics 111* (1993) 409-420.

[158] M.W. Padberg, Almost Perfect Matrices and Graphs, manuscript, Forschungsinstitut für Diskrete Mathematik, Universität Bonn, Germany (1998).

[159] B.A. Papernov, Feasibility of Multicommodity Flows, in: *Studies in Discrete Optimization* (A.A. Friedman ed.), Nauka, Moscow (1976) 230-261 (in Russian).

[160] K.R. Parthasarathy and G. Ravindra, The Strong Perfect Graph Conjecture is True for $K_{1,3}$-free Graphs, *Journal of Combinatorial Theory B 21* (1976) 212-223.

[161] K.R. Parthasarathy and G. Ravindra, The Validity of the Strong Perfect Graph Conjecture for $K_4 \setminus \{e\}$-free Graphs, *Journal of Combinatorial Theory B 26* (1979) 98-100.

[162] W.R. Pulleyblank, personal communication (1992).

[163] L. Qi, On the Set Covering Polytope and the MFMC-Clutter, preprint, School of Mathematics, University of New South Wales (1989).

[164] N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, The Four-Colour Theorem, *Journal of Combinatorial Theory B 70* (1997) 2-44.

[165] N. Robertson, P. D. Seymour and R. Thomas, Tutte's edge-coloring conjecture, *Journal of Combinatorial Theory B 70* (1997) 166-183.

[166] A. van Rooij and H. Wilf, The Interchange Graph of a Finite Graph, *Acta Mathematica Academiae Scientiarum Hungaricae 16* (1965) 263-269.

[167] D.J. Rose, R.E. Tarjan and G.S. Lueker, Algorithmic Aspects of Vertex Elimination on Graphs, *SIAM Journal on Computing 5* (1976) 266-283.

[168] H. Sachs, On the Berge Conjecture Concerning Perfect Graphs, in *Combinatorial Structures and their Applications*, R. Guy, H. Hanani, N. Sauer and J. Schönheim eds., Gordon and Breach, New York (1970) 377-384.

[169] D. P. Sanders, P. D. Seymour and R. Thomas, Edge Three-Coloring Cubic Doublecross Graphs, in preparation.

[170] D. P. Sanders and R. Thomas, Edge Three-Coloring Cubic Apex Graphs, manuscript.

[171] A. Schrijver, A Counterexample to a Conjecture of Edmonds and Giles, *Discrete Mathematics 32* (1980) 213-214.

[172] A. Schrijver, Min-Max Results in Combinatorial Optimization, in *Mathematical Programming The State of the Art Bonn 1982*, A. Bachem, M. Grötschel, B. Korte eds., Springer-Verlag (1983) 439-500.

[173] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York (1986).

[174] A. Sebö, A Quick Proof of Seymour's Theorem on T-joins, *Discrete Mathematics 64* (1987) 101-103.

[175] A. Sebö, The Schrijver-System of Odd-Join Polyhedra, *Combinatorica 8* (1988) 103-116.

[176] A. Sebö, A Very Short Proof of Seymour's Theorem on t-Joins, preprint, IMAG, Grenoble, France (1990).

[177] A. Sebő, The connectivity of minimal imperfect graphs, *Journal of Graph Theory* 23 (1996) 77-85.

[178] A. Sebö, On Critical Edges in Minimal Imperfect Graphs, *Journal of Combinatorial Theory B* (1996) 62-85.

[179] E.C. Sewell, Binary Integer Programs with Two Variables per Inequality, *Mathematical Programming 75* (1996) 467-476.

[180] P.D. Seymour, A Forbidden Minor Characterization of Matroid Ports, *Quaterly Journal of Mathematics Oxford (2) 27* (1976) 407-413.

[181] P.D. Seymour, The Forbidden Minors of Binary Clutters, *Journal London Math. Soc. 12* (1976) 356-360.

[182] P.D. Seymour, A Note on the Production of Matroid Minors, *Journal of Combinatorial Threory B 22* (1977) 289-295.

[183] P.D. Seymour, The Matroids with the Max-Flow Min-Cut Property, *Journal of Combinatorial Theory B 23* (1977) 189-222.

[184] P.D. Seymour, A Two-Commodity Cut Theorem, *Discrete Mathematics 23* (1978) 177-181.

[185] P.D. Seymour, On Multi-Colourings of Cubic Graphs and Conjectures of Fulkerson and Tutte, *Proceedings of the London Mathematical Soceity (3) 42* (1979) 423-460.

[186] P.D. Seymour, Decomposition of Regular Matroids, *Journal of Combinatorial Theory B 28* (1980) 305-359.

[187] P.D. Seymour, Matroids and Multicommodity Flows, *European Journal of Combinatorics 2* (1981) 257-290.

[188] P.D. Seymour, On Odd Cuts and Plane Multicommodity Flows, *Proceedings of the London Mathematical Society 42* (1981) 178-192.

[189] P.D. Seymour, On Lehman's Width-Length Characterization, in: *Polyhedral Combinatorics* (W. Cook and P.D. Seymour eds.), *DIMACS Series in Discrete Mathematics and Theoretical Computer Science 1*, American Mathematical Society, Providence, R.I. (1990) 107-117.

[190] P. G. Tait, Note on a Theorem in Geometry of Position, *Trans. Roy. Soc. Edinburgh 29* (1880) 657-660.

[191] K. Truemper, Complement Total Unimodularity, *Linear Algebra and its Applications 30* (1930) 77-92.

[192] K. Truemper, Alfa-Balanced Graphs and Matrices and GF(3)-Representability of Matroids, *Journal of Combinatorial Theory B 32* (1982) 112-139.

[193] K. Truemper, Polynomial Theorem Proving I. Central Matrices, *Technical Report UTDCS 34-90* (1990).

[194] K. Truemper, A Decomposition Theory for Matroids. VII. Analysis of Minimal Violation Matrices, *Journal of Combinatorial Theory (B) 55* (1992) 302-335.

[195] K. Truemper, *Matroid Decomposition*, Academic Press, Boston (1992).

[196] K. Truemper, *Effective Logic Computation*, Wiley, New York (1998).

[197] K. Truemper, R. Chandrasekaran, Local Unimodularity of Matrix-Vector Pairs, *Linear Algebra and its Applications 22* (1978) 65-78.

[198] F.T. Tseng and K. Truemper, A Decomposition of Matroids with the Max-Flow Min-Cut Property, *Discrete Applied Mathematics 15* (1986) 329-364. Addendum: *Discrete Applied Mathematics 20* (1988) 87-88.

[199] A. Tucker, The validity of the Perfect Graph Conjecture for K_4-free Graphs, in *Topics on Perfect Graphs*, Berge and Chvátal eds., *Annals of Discrete Mathematics* 21 (1984) 149-157.

[200] W.T. Tutte, On Hamiltonian Circuits, *J. London Math. Soc. 21* (1946) 98-101.

[201] W.T. Tutte, The Factorization of Linear Graphs, *J. London Math. Soc. 22* (1947) 107-111.

[202] W.T. Tutte, A Homotopy Theorem for Matroids I, II, *Transactions of the American Methematical Society 88* (1958) 144-160, 161-174.

[203] W.T. Tutte, On the Algebraic Theory of Graph Colorings, *Journal of Combinatorial Theory 1* (1966) 15-50.

[204] R.F. Veinott, G.B. Dantzig, Integral Extreme Points, *SIAM Review 10* (1968) 371-372.

[205] V.G. Vizing, The Chromatic Class of a Multigraph, *Kibernetika 3* (1965) 29-39 (in Russian).

[206] K. Vušković, *Holes in Bipartite Graphs*, PhD dissertation, Carnegie Mellon University (1994).

[207] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann. 114* (1937) 570-590.

[208] D.B. West, *Introduction to Graph Theory*, Prentice Hall (1996).

[209] S. Whitesides, An Algorithm for Finding Clique Cutsets, *Information Processing Letters 12* (1981) 31-32.

[210] D.R. Woodall, Menger and König Systems, *Lecture Notes in Mathematics 624*, Springer-Verlag (1978) 620-635.

[211] M. Yannakakis, On a Class of Totally Unimodular Matrices, *Mathematics of Operations Research 10* (1985) 280-304.

Index