1 Packing and Covering

Clutters

Hypergraph: A *hypergraph* $\mathcal{H}$ consists of a finite set of *vertices*, denoted $V(\mathcal{H})$, a finite set of *edges*, denoted $E(\mathcal{H})$, and an incidence relation on $V(\mathcal{H}) \times E(\mathcal{H})$. We associate an edge $e \in E(\mathcal{H})$ with the set of vertices it is incident with, so for instance, we write $e = \{a, b, c\}$ if e is incident with precisely the vertices a, b, c.

Matching: A *matching* in $\mathcal{H}$ is a collection of edges so that no vertex is incident with more than one. Informally, packing is the problem of finding the largest matching.

Cover: A *cover* in $\mathcal{H}$ is a collection of edges so that every vertex is incident with at least one. Informally, covering is the problem of finding the smallest cover.

Maximal & Minimal: We will frequently consider subsets of a set X with a certain property $\star$. We say that $A \subseteq X$ is *minimal (maximal)* with property $\star$ if A has property $\star$ but no proper subset (superset) of A does.

Clutter: A *clutter* $\mathcal{C}$ is a hypergraph with the property that whenever e, f are distinct edges $e \not\subseteq f$. Note that if $\mathcal{H}$ is a hypergraph, then the minimal nonempty edges of $\mathcal{H}$ form a clutter $\mathcal{C}_1$ and the maximal edges of $\mathcal{H}$ form a clutter $\mathcal{C}_2$. Furthermore, packing problems in $\mathcal{H}$ reduce to packing problems in $\mathcal{C}_1$ while covering problems in $\mathcal{H}$ reduce to covering problems in $\mathcal{C}_2$.

Incidence Matrix: The *incidence matrix* or *clutter matrix* of $\mathcal{C}$, denoted $M(\mathcal{C})$, is the matrix indexed by $E(C) \times V(C)$ with a 1 in position e, v if e and v are incident and a 0 otherwise.

Packing

Packing: We let $\nu(\mathcal{C})$ denote the size of the largest matching in $\mathcal{C}$.

Transversal: A *transversal* of $\mathcal{C}$ is a subset $X \subseteq V(\mathcal{C})$ with the property that $X \cap e \neq \emptyset$ for every $e \in E(\mathcal{C})$. We let $\tau(\mathcal{C})$ denote the size of the smallest transversal in $\mathcal{C}$

Observation 1.1 $\nu(\mathcal{C}) \leq \tau(\mathcal{C})$.

Pack: We say that $\mathcal{C}$ *packs* if $\nu(\mathcal{C}) = \tau(\mathcal{C})$.

Blocker: The *blocker* of $\mathcal{C}$, denoted $b(\mathcal{C})$, is a clutter with vertex set $V(\mathcal{C})$, where e is an edge of $b(\mathcal{C})$ if e is a minimal transversal of $\mathcal{C}$.

Proposition 1.2 *$b(b(\mathcal{C})) = \mathcal{C}$ for every clutter $\mathcal{C}$.*

Proof: Let $V = V(\mathcal{C})$. We first prove two easy properties:

(1) Every $e \in E(\mathcal{C})$ contains an edge of $b(b(\mathcal{C}))$.

By definition, every edge in $b(\mathcal{C})$ intersects e, so e is a transversal of $b(\mathcal{C})$. It therefore contains a minimal transversal of $b(\mathcal{C})$, or equivalently, an edge of $b(b(\mathcal{C}))$.

(2) Every $f \in E(b(b(\mathcal{C})))$ contains an edge of $\mathcal{C}$.

Suppose (for a contradiction) that (2) is false. Then the set $V \setminus f$ must intersect every edge in $E(\mathcal{C})$, so $V \setminus f$ is a transversal of $\mathcal{C}$. Thus $V \setminus f$ contains a minimal transversal g of $\mathcal{C}$ which is, by definition, an edge of $b(\mathcal{C})$. However, this is a contradiction, as f is a transversal of $b(\mathcal{C})$, but $f \cap g = \emptyset$.

To finish the proof, let e be an edge of $\mathcal{C}$. Then by (1), e must contain an edge f of $b(b(\mathcal{C}))$, and by (2), f must contain an edge e' of $\mathcal{C}$. But then we have $e' \subseteq f \subseteq e$ and it then follows from the definition of clutter that $e = f = e'$. Since $e = f$, we have now shown that e is also an edge of $b(b(\mathcal{C}))$. By a similar argument we find that every edge of $b(b(\mathcal{C}))$ is also an edge of $\mathcal{C}$, so $\mathcal{C} = b(b(\mathcal{C}))$ as desired. $\square$

Examples. Fix a simple connected graph G and consider the following clutters.

(i) $\mathcal{C}_1 = G$ (so $V(\mathcal{C}_1) = V(G)$, $E(\mathcal{C}_1) = E(G)$ and we have the same incidence relation as in G). Then (check!) the edges of $b(\mathcal{C}_1)$ are the minimal vertex covers of G. Note that a theorem of König shows that $\mathcal{C}_1$ packs whenever G is bipartite.

(ii) $\mathcal{C}_2$ has vertex set $E(G)$, and $E(\mathcal{C}_2) = \{E(T) : T \text{ is a spanning tree of } G\}$. Then (check!) the edges of $b(\mathcal{C}_2)$ are precisely the minimal edge cuts of G.

(iii) $\mathcal{C}_3$ has vertex set $E(G)$, and $E(\mathcal{C}_3) = \{E(C) : C \text{ is a cycle of } G\}$. Then (check!) the edges of $b(\mathcal{C}_3)$ are precisely the complements of spanning trees of G.

(iv) Fix $s, t \in V(G)$ and let $\mathcal{C}_4$ be the clutter with $V(\mathcal{C}_4) = E(G)$ and $E(\mathcal{C}_4) = \{E(P) : P \text{ is an } st\text{-path}\}$. Then (check!) the edges of $b(\mathcal{C}_4)$ are precisely the minimal *st* edge cuts of G. Note that Menger's theorem on edge-connectivity shows that $\mathcal{C}_4$ packs.

Weightings: Let $V = V(\mathcal{C})$ and let $\mathbb{Z}_+$ denote the set of nonnegative integers. We will call an element $w \in \mathbb{Z}_+^V$ a *weighting* and view it both as a function and as a vector. If $S \subseteq V$ we let $w(S) = \sum_{v \in S} w(v)$. We define the weighted packing parameters τ_w and ν_w as follows (were the clutter not clear from context we would write $\tau_w(\mathcal{C})$ and $\nu_w(\mathcal{C})$.

$$\tau_w = \min\{w(e) : e \in E(b(\mathcal{C}))\}$$
$$\nu_w = \max\{|\mathcal{F}| : \mathcal{F} \text{ a multiset of edges of } \mathcal{C} \text{ s.t. every } v \in V \text{ is used} \leq w(v) \text{ times}\}$$

Observation 1.3 *$\nu_w(\mathcal{C}) \leq \tau_w(\mathcal{C})$.*

MFMC: We say that $\mathcal{C}$ has the *MFMC property* (MFMC is an abbreviation for Max-Flow Min-Cut) or just that $\mathcal{C}$ is *MFMC* if $\tau_w = \nu_w$ for every $w \in \mathbb{Z}_+^V$.

Theorem 1.4 (Ford-Fulkerson) *Let D be a digraph with vertices s, t and let $\mathcal{C}$ be the clutter with vertex set $E(D)$ whose edges are the collection of all directed st-paths in D. Then $\mathcal{C}$ has the MFMC property.*

Covering

Note: If $\mathcal{C}$ is a clutter which has a vertex not incident with any edge, then it has no cover. To avoid this annoyance, we shall henceforth assume that our clutters do not have such vertices.

Covering: We let $\kappa(\mathcal{C})$ denote the size of the smallest cover in $\mathcal{C}$.

Independent Sets: A subset $X \subseteq V(\mathcal{C})$ is *independent* if $|X \cap e| \leq 1$ for every edge $e \in E(\mathcal{C})$. The size of the largest independent set is denoted $\alpha(\mathcal{C})$.

Observation 1.5 *$\kappa(\mathcal{C}) \geq \alpha(\mathcal{C})$ for every clutter $\mathcal{C}$.*

Covers: We say that the clutter $\mathcal{C}$ *covers* if $\kappa(\mathcal{C}) = \alpha(\mathcal{C})$.

Antiblocker: The *antiblocker* of $\mathcal{C}$, denoted $a(\mathcal{C})$, is a clutter with vertex set $V(\mathcal{C})$, where e is an edge of $a(\mathcal{C})$ if e is a maximal independent set.

Helly Property: If $x, y \in V(\mathcal{C})$ write $x \sim y$ if there exists $e \in E(\mathcal{C})$ with $x, y \in e$. We say that $\mathcal{C}$ has the *Helly Property* if every subset $X \subseteq V(\mathcal{C})$ for which $x \sim y$ for all $x, y \in X$ has the property that there is an edge containing X.

Clique-Node & Independence-Node Clutters: The *clique-node (independence-node)* clutter of a graph G is the clutter $CN(G)$ ($IN(G)$) with vertex set $V(G)$ and with an edge e whenever $e \subseteq V(G)$ is a maximal clique (independent set) of G.

Observation 1.6

(i) $IN(G) = CN(\bar{G})$

(ii) *$IN(G)$ and $CN(G)$ are antiblockers of one another.*

(iii) *$IN(G)$ covers if and only if $\omega(G) = \chi(G)$.*

Proposition 1.7 *Let $\mathcal{C}$ be a clutter. Then the following are equivalent.*

(i) $a(a(\mathcal{C})) = \mathcal{C}$

(ii) *$\mathcal{C}$ has the Helly Property.*

(iii) *$\mathcal{C} = CN(G)$ for some graph G.*

Proof: Homework.

Weighting: For $w \in \mathbb{Z}_+^V$ we define κ_w and α_w as follows (as usual, we write $\kappa_w(\mathcal{C})$ and $\alpha_w(\mathcal{C})$ if the clutter is not clear from context):

$$\alpha_w = \max\{w(e) : e \in E(a(\mathcal{C}))\}$$
$$\kappa_w = \min\{|(\mathcal{F}| : \mathcal{F} \text{ a multiset of edges of } \mathcal{C} \text{ s.t. every } v \in V \text{ is used } \geq w(v) \text{ times}\}$$

Observation 1.8 $\kappa_w(\mathcal{C}) \geq \alpha_w(\mathcal{C})$.

Perfect$^+$: We say that $\mathcal{C}$ is *Perfect$^+$* if $\alpha_w(\mathcal{C}) = \kappa_w(\mathcal{C})$ for every $w \in \mathbb{Z}_+^V$.