2 Polyhedra

Preliminaries

By default, we treat elements of $\mathbb{R}^m$ as column vectors and if $x, y \in \mathbb{R}^m$ we let $x \cdot y = x^\top y$.

Polyhedron: A *polyhedron* $P \subseteq \mathbb{R}^n$ is any set of the form $\{x \in \mathbb{R}^n : Ax \leq b\}$ where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. If $c \in \mathbb{R}^n$ and $x \cdot c \geq t$ for every $x \in P$ then $\{x \in P : x \cdot c = t\}$ is a *face* of P. A *vertex* of P is a face which consists of a single point.

Observation 2.1 *If x is a vertex of the polyhedron P, then there exists $w \in \mathbb{Z}^n$ and $\lambda \in \mathbb{R}$ so that $\{x\} = \{y \in P : y^\top w \geq \lambda\}$.*

Proof: Choose a vector u and a number μ so that $\{x\} = \{y \in P : y^\top u \geq \mu\}$. It follows from the density of the rationals that we may assume u is rational. Let N be the least common multiple of the denominators which appear in the entries of the vector u. Now setting $w = Nu$ and $\lambda = N\mu$ we have that $\{x\} = \{y \in P : y^\top w \geq \lambda\}$ as required. □

Pointed: We say that P is *pointed* if every minimal face is a vertex. Note that if $P \subseteq \mathbb{R}^m_+$ then P must be pointed.

Integral: A point $x \in \mathbb{R}^m$ is *integral* if every coordinate is an integer. A polyhedron $P \subseteq \mathbb{R}^m$ is *integral* if every minimal face contains an integral point. Note that a pointed polyhedron is integral if and only if every vertex is integral.

Convex Hull: The *convex hull* of a set $X \subseteq \mathbb{R}^m$ is the unique minimal convex set which includes X. Note that the *convex hull* of the rows of A is precisely

$$\{y^\top A : y \in \mathbb{R}^m_+ \text{ and } y \cdot 1 = 1\}.$$

Up & Down-Monotone: We say that a set $X \subseteq \mathbb{R}^m$ is *up-monotone* if whenever $x \in X$ and $y \geq x$ we have $y \in X$. Similarly, we say that X is *down monotone in* $\mathbb{R}^m_+$ if whenever $x \in X$ and $0 \leq y \leq x$ we have $y \in X$.

Up & Down-Hull: If $X \subseteq \mathbb{R}^m_+$ the *up-hull* of X is the unique minimal up-monotone set which includes X. Similarly, the *down-hull of X in* $\mathbb{R}^m_+$ is the unique minimal down-monotone set in $\mathbb{R}^m_+$ which includes X.

Essential Vertex: If $P \subseteq \mathbb{R}^m_+$ is down monotone in $\mathbb{R}^m_+$ then a vertex x of P is *essential* if x is not contained in the down-hull of $P \setminus x$.

Observation 2.2 *Let $P \subseteq \mathbb{R}^m_+$ be a polyhedron. Then*

(i) *If P is bounded, then P is the convex hull of its vertices.*

(ii) *If P is up-monotone, then P is the up-hull of its vertices.*

(iii) *If P is down-monotone in $\mathbb{R}^m_+$, then P is the down-hull of its essential vertices.*

Up & Down: Let $\mathcal{C}$ be a clutter and let $M = M(\mathcal{C})$. We define the polyhedra $Up(\mathcal{C})$ to be the up-hull of the incidence vectors of the edges of $\mathcal{C}$ and $Down(\mathcal{C})$ to be the down-hull of the incidence vectors of the edges of $\mathcal{C}$ in $\mathbb{R}^V_+$. Thus, we have:

$$Up(\mathcal{C}) = \{x \in \mathbb{R}^m_+ : x^\top \geq y^\top A \text{ for some } y \in \mathbb{R}^m_+ \text{ with } y \cdot 1 = 1\}$$
$$Down(\mathcal{C}) = \{x \in \mathbb{R}^m_+ : x^\top \leq y^\top A \text{ for some } y \in \mathbb{R}^m_+ \text{ with } y \cdot 1 = 1\}$$

Proposition 2.3 *Let $\mathcal{C} = (V, E)$ be a clutter and let $X \subseteq \mathbb{R}^V_+$ be the set of incidence vectors of its edges. Then X is the set of vertices of $Up(\mathcal{C})$ and the set of essential vertices of $Down(\mathcal{C})$.*

Proof: Homework.

The Packing Polyhedron and Ideal Clutters

Blocking Polyhedra: If $P \subseteq \mathbb{R}^S_+$ is an up-monotone polyhedron, its *blocker* is

$$b(P) = \{y \in \mathbb{R}^S_+ : x^\top y \geq 1 \text{ for every } x \in P\}$$

Lemma 2.4 *Let $P \subseteq \mathbb{R}^S_+$ be an up-monotone polyhedron.*

(i) *if $z \notin P$ there exist $y \in \mathbb{R}^S_+$ and $\lambda \in \mathbb{R}_+$ s.t. $y \top z < \lambda$ and $y \top x \geq \lambda$ for every $x \in P$.*

(ii) *if $x \in P$ is a vertex, there exist $y \in \mathbb{R}^S_+$ and $\lambda \in \mathbb{R}_+$ s.t. $\{x\} = \{z \in P : y^\top z \leq \lambda\}$.*

Proof: (i): Since P is closed and convex and $z \notin P$, we may choose a hyperplane H which separates z and P. Choose a normal vector y for H and suppose (for a contradiction) that there are indices i, j with $y_i > 0$ and $y_j < 0$. Choose a point $x \in P$ and suppose that $y^\top x = t$. If we start at x and then increase the coordinate x_i (x_j) we stay in P, and it follows that there are points in P whose dot product with y is equal to t' for any $t' \geq t$ ($t' \leq t$). But this contradicts the assumption that H is disjoint from P. So, by possibly replacing y by $-y$ we may assume that $y \geq 0$. Choose $\lambda \neq 0$ so that $H = \{x \in \mathbb{R}^S_+ : x^\top y = \lambda$. It now follows from the fact that y, z, P all lie in the nonnegative orthant that $\lambda > 0$. This completes the proof. The proof of (ii) is similar except that H is chosen to be a hyperplane which intersects P only at x. □

Proposition 2.5 *$b(b(P)) = P$ for every up-monotone polyhedron $P \subseteq \mathbb{R}^S_+$.*

Proof: Every $x \in P$ satisfies $x \geq 0$ and $x^\top y \geq 1$ for every $y \in b(P)$ by definition. It follows from this that $x \in b(b(P))$, so $P \subseteq b(b(P))$.

Suppose (for a contradiction) that $b(b(P)) \not\subseteq P$, and choose $z \in b(b(P)) \setminus P$. Apply the lemma to choose a vector $y \in \mathbb{R}^S_+$ and $\lambda \in \mathbb{R}_+$. Now, replacing y by the vector $\frac{1}{\lambda} y$ gives us $y^\top z < 1$ and $y^\top x \geq 1$ for all $x \in P$. It follows from the latter that $y \in b(P)$, but then the former condition contradicts $z \in b(b(P))$. □

Let $\mathcal{C}$ be a clutter with $V = V(\mathcal{C})$, set $A = M(\mathcal{C})$ and $B = M(b(\mathcal{C}))$.

Fractional Transversal: A *fractional transversal* of $\mathcal{C}$ is a vector $x \in \mathbb{R}^V_+$ with the property that $Ax \geq 1$. So, in other words, a fractional transversal is a weighting of the vertices with the property that every edge gets a total weight of ≥ 1. Note that a $0, 1$ vector is a fractional transversal if and only if it is the incidence vector of a transversal.

Packing Polyhedron: We define the *packing polyhedron* as follows:

$$Pack(\mathcal{C}) = \{x \in \mathbb{R}^V_+ : Ax \geq 1\}$$

So, in words, the packing polyhedron is the set of all fractional transversals.

Ideal: The clutter $\mathcal{C}$ is *ideal* if $Pack(\mathcal{C})$ is integral.

Observation 2.6

(i) $Up(b(\mathcal{C})) \subseteq Pack(\mathcal{C})$

(ii) $Up(b(\mathcal{C})) = Pack(\mathcal{C})$ *if and only if* $\mathcal{C}$ *is ideal.*

Proof: Part (i) follows immediately from the observation that each incidence vector of an edge in $b(\mathcal{C})$ is contained in $Pack(\mathcal{C})$. Part (ii) follows from the observation that every integral point in $Pack(\mathcal{C})$ is also contained in $Up(b(\mathcal{C}))$. $\square$

Observation 2.7 $b(Up(\mathcal{C})) = Pack(\mathcal{C})$

Proof:

$$\begin{aligned} b(Up(\mathcal{C})) &= \{y \in \mathbb{R}^V_+ :\ y^\top x \geq 1 \text{ for every } x \in Up(\mathcal{C})\ \} \\ &= \{y \in \mathbb{R}^V_+ :\ y^\top x \geq 1 \text{ whenever } x \text{ is the incidence vector of an edge of } \mathcal{C}\ \} \\ &= Pack(\mathcal{C}) \qquad \square \end{aligned}$$

For the remaining two results in this section, let us name the following polyhedra:

$$\begin{aligned} P &= Pack(\mathcal{C}) \\ Q &= Pack(b(\mathcal{C})) \\ P_I &= Up(b(\mathcal{C})) \\ Q_I &= Up(\mathcal{C}) \end{aligned}$$

So, by the previous observation $b(P) = Q_I$ and $b(Q) = P_I$. Observation 2.6 shows that $P_I \subseteq P$ and $Q_I \subseteq Q$ and further that $\mathcal{C}$ is ideal if and only if $P = P_I$ and $b(\mathcal{C})$ is idea if and only if $Q = Q_I$.

Theorem 2.8 (Lehman) $\mathcal{C}$ *is ideal if and only if* $b(\mathcal{C})$ *is ideal.*

Proof:

$$\begin{aligned} \mathcal{C} \text{ is ideal} &\Leftrightarrow P = P_I \\ &\Leftrightarrow b(P) = b(P_I) \\ &\Leftrightarrow Q_I = Q \\ &\Leftrightarrow b(\mathcal{C}) \text{ is ideal.} \end{aligned}$$

$\square$

Theorem 2.9 (Lehman's width-length inequality) *The following are equivalent*

(i) $\mathcal{C}$ *and* $b(\mathcal{C})$ *are ideal.*

(ii) $\left(\min_{e\in E(\mathcal{C})} w(e)\right)\left(\min_{f\in E(b(\mathcal{C}))} \ell(f)\right) \leq w^\top \ell$ *for every* $w, \ell \in \mathbb{R}^V_+$.

(here we let $x(S) = \sum_{s\in S} x(s)$ *whenever* $x \in \mathbb{R}^V$ *and* $S \subseteq V$*).*

Proof: The result is trivial whenever $\min_{e\in E(\mathcal{C})} w(e) = 0$ or $\min_{f\in E(b(\mathcal{C}))} \ell(f) = 0$. Furthermore, it is invariant under scaling w or ℓ by a positive constant, so we may assume that $\min_{e\in E(\mathcal{C})} w(e) = 1 = \min_{f\in E(b(\mathcal{C}))} \ell(f)$. We now have the following (the first identity is a consequence of these assumptions).

$$
\begin{aligned}
(ii) &\Leftrightarrow w^\top \ell \geq 1 \text{ whenever } w \in P \text{ and } \ell \in Q \\
&\Leftrightarrow P \subseteq b(Q) \text{ and } Q \subseteq b(P) \\
&\Leftrightarrow P \subseteq P_I \text{ and } Q \subseteq Q_I \\
&\Leftrightarrow P = P_I \text{ and } Q = Q_I \\
&\Leftrightarrow \mathcal{C} \text{ and } b(\mathcal{C}) \text{ are ideal} \\
&\Leftrightarrow (i) \qquad \square
\end{aligned}
$$

The Covering Polyhedron and Perfect Clutters

Antiblocker: If $P \subseteq \mathbb{R}^S_+$ is a down-monotone polyhedron, its *antiblocker* is

$$a(P) = \{y \in \mathbb{R}^S_+ : x^\top y \leq 1 \text{ for every } x \in P\}$$

Lemma 2.10 *Let* $P \subseteq \mathbb{R}^S_+$ *be a down-monotone polyhedron.*

(i) *if* $z \notin P$ *there exist* $y \in \mathbb{R}^S_+$ *and* $\lambda \in \mathbb{R}_+$ *s.t.* $y\top z > \lambda$ *and* $y\top x \leq \lambda$ *for every* $x \in P$.

(ii) *if* $x \in P$ *is an essential vertex, there exist* $y \in \mathbb{R}^S_+$ *and* $\lambda \in \mathbb{R}_+$ *s.t.* $\{x\} = \{z \in P : y^\top z \geq \lambda\}$.

Proof: Homework

Theorem 2.11 $a(a(P)) = P$ *for every down-monotone polyhedron* $P \subseteq \mathbb{R}^S_+$.

Proof: Homework

Let $\mathcal{C}$ be a clutter with $V = V(\mathcal{C})$ and set $A = M(\mathcal{C})$.

Fractional Independent Set: A *fractional independent set* of $\mathcal{C}$ is a vector $x \in \mathbb{R}^V_+$ with the property that $Ax \leq 1$. So, in other words, a fractional independent set is a weighting of the vertices with the property that every edge gets a total weight of ≤ 1. Note that a $0, 1$ vector is a fractional independent set if and only if it is the incidence vector of an independent set.

Covering Polyhedron: We define the *covering polyhedron* as follows:

$$Cov(\mathcal{C}) = \{x \in \mathbb{R}^V_+ : Ax \leq 1\}$$

So, in words, the covering polyhedron is the set of all fractional independent sets.

Perfect: The clutter $\mathcal{C}$ is *perfect* if $Cov(\mathcal{C})$ is integral.

Observation 2.12

(i) $Down(a(\mathcal{C})) \subseteq Cov(\mathcal{C})$

(ii) $Down(a(\mathcal{C})) = Cov(\mathcal{C})$ *if and only if* $\mathcal{C}$ *is perfect.*

Proof: Part (i) follows immediately from the observation that each incidence vector of an edge in $a(\mathcal{C})$ is contained in $Cov(\mathcal{C})$. Part (ii) follows from the observation that every integral point in $Cov(\mathcal{C})$ is also contained in $Down(a(\mathcal{C}))$. $\square$

Observation 2.13 $a(Down(\mathcal{C})) = Cov(\mathcal{C})$.

Proof:

$$\begin{aligned} a(Down(\mathcal{C})) &= \{y \in \mathbb{R}^V_+ : y^\top x \leq 1 \text{ for every } x \in Down(\mathcal{C})\} \\ &= \{y \in \mathbb{R}^V_+ : y^\top x \leq 1 \text{ whenever } x \text{ is the incidence vector of an edge in } \mathcal{C}\} \\ &= Cov(\mathcal{C}) \qquad \square \end{aligned}$$

Define the following polyhedra:

$$\begin{aligned} P &= Cov(\mathcal{C}) \\ Q &= Cov(a(\mathcal{C})) \\ R_I &= Down(\mathcal{C}) \\ P_I &= Down(a(\mathcal{C})) \\ Q_I &= Down(a(a(\mathcal{C})) \end{aligned}$$

So, by the previous observation, $a(P) = R_I$ and $a(Q) = P_I$. It is immediate from the definitions that $R_I \subseteq Q_I$ and Observation 2.12 shows that we have $P_I \subseteq P$ and $R_I \subseteq Q_I \subseteq Q$. Further, this observation shows that $\mathcal{C}$ is perfect if and only if $P = P_I$ and $a(\mathcal{C})$ is perfect if and only if $Q = Q_I$.

Theorem 2.14 *If $\mathcal{C}$ is perfect, then $a(\mathcal{C})$ is perfect.*

Proof: By our observations

$$\mathcal{C} \text{ perfect } \Rightarrow\ P = P_I \ \Rightarrow\ a(P) = a(P_I) \ \Rightarrow R_I = Q \ \Rightarrow\ Q_I = Q \ \Rightarrow\ a(\mathcal{C}) \text{ perfect.} \qquad \square$$

Theorem 2.15 (Chvátal) *If $\mathcal{C}$ is perfect, then*

(i) $a(a(\mathcal{C})) = \mathcal{C}$

(ii) $\mathcal{C} = CN(G)$ *for a graph G.*

Proof: For (i) we have

$$\mathcal{C} \text{ perfect } \Rightarrow\ P = P_I \ \Rightarrow\ a(P) = a(P_I) \ \Rightarrow R_I = Q \ \Rightarrow\ R_I = Q_I \ \Rightarrow\ \mathcal{C} = a(a(\mathcal{C})).$$

The proof of (ii) follows from this and an earlier exercise. $\square$

Theorem 2.16 *The following are equivalent.*

(i) *$\mathcal{C}$ is perfect.*

(ii) $\left(\max_{e \in E(\mathcal{C})} w(e)\right) \left(\max_{f \in E(a(\mathcal{C}))} \ell(f)\right) \geq w^\top \ell$ *for every* $w, \ell \in \mathbb{R}^V_+$.

Proof: Homework.