3 Linear Programming

LP Duality

Vectors in this section are column vectors by default. The dimensions of our vectors are frequently not stated, but must be inferred from context. If a, b are vectors from the same space, we write $a \leq b$ if $a_i \leq b_i$ for every coordinate i. Similarly, we write $a \geq 0$ if a is coordinatewise greater than the vector of zeros.

Cone: A set $C \subseteq \mathbb{R}^n$ is a *cone* if $\lambda x \in C$ whenever $x \in C$ and $\lambda \geq 0$.

Polyhedral Cone: A *polyhedral cone* is any set of the form $\{Ax : x \geq 0\}$ where A is a real $m \times n$ matrix.

Lemma 3.1 (Farkas Lemma) *If A is an $m \times n$ real matrix and $b \in \mathbb{R}^m$, then exactly one of the following holds:*

(i) *There exists $x \geq 0$ so that $Ax = b$.*

(ii) *There exists y so that $y^\top A \geq 0$ and $y^\top b < 0$.*

Note: Lemma 3.1 is equivalent to the obvious fact that given a point b and a cone $C = \{Ax : x \geq 0\}$, either (i) $b \in C$ or (ii) there is a hyperplane (with normal y) through the origin separating b from C.

Hint for Proof: It is immediate that (i) and (ii) are mutually exclusive as otherwise we would have $0 > y^\top b = y^\top Ax \geq 0$ which is contradictory. Now, assume that (i) does not hold. It then follows from the fact that C is closed and convex that there is a hyperplane H which separates b from C. Shift H to a parallel hyperplane H' (keeping the same normal vector) until it meets the cone C. Since 0 is in every minimal face of C, it follows that $0 \in H'$. Now by possibly replacing y by $-y$ we may arrange that $y^\top b < 0$ and $y^\top A \geq 0$.

Corollary 3.2 *If A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, then exactly one of the following holds:*

(i) *There exists x so that $Ax \leq b$.*

(ii) *There exists $y \geq 0$ so that $y^\top A = 0$ and $y^\top b < 0$.*

Proof: It is immediate that (i) and (ii) are mutually exclusive, as otherwise we would have $0 = y^\top Ax \leq y^\top b < 0$ which is contradictory.

To see that one of these conclusions must hold, consider the matrix $A' = [I \; A \; -A]$ and apply the Farkas Lemma to A' and b. If there exists a vector $z^\top = [w^\top, x_p^\top, x_m^\top] \geq 0$ so that $A'z = b$, then we have that $A(x_p - x_m) \leq b$, so (i) holds. Otherwise, there must be a vector y so that $y^\top A' \geq 0$ and $y^\top b < 0$, but then $y \geq 0$ and $y^\top A = 0$ so (ii) holds. $\square$

Linear Programming: Fix an $m \times n$ matrix A and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. A *linear program* and the associated *dual* are given as follows:

LP (primal)	Dual
maximize $c^\top x$	minimize $y^\top b$
s.t. $x \geq 0$	s.t. $y \geq 0$
$Ax \leq b$	$y^\top A \geq c$

We say that a point x (y) satisfying $x \geq 0$ and $Ax \leq b$ ($y \geq 0$ and $y^\top A \geq c$) is a *feasible* point for the linear program (dual). If no such point exists the problem is called *infeasible*. If the primal (dual) problem is feasible but has no maximum (minimum), it is called *unbounded*.

Observation 3.3 (Weak Duality) *If x is feasible for the Linear Program and y is feasible for the dual, then*

$$c^\top x \leq y^\top b$$

Proof: $c^\top x \leq (y^\top A)x = y^\top (Ax) \leq y^\top b$ $\square$

Note: It follows from the above that any feasible point in the dual gives an upper bound on the primal problem (and vice versa). So, in particular, if the dual problem is feasible, then the primal problem is bounded.

Theorem 3.4 (Strong Duality) *If the primal and dual problem are feasible, then the optimum points x, y satisfy $c^\top x = y^\top b$.*

Proof: Consider the following equation

$$\begin{bmatrix} -I & 0 \\ A & 0 \\ -c^\top & b^\top \\ 0 & -I \\ 0 & -A^\top \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} 0 \\ b \\ 0 \\ 0 \\ -c \end{bmatrix}$$

If there exist x, y satisfying the above equation, then $x \ge 0$, $Ax \le b$ so x is feasible, $y \ge 0$ and $y^\top A \ge c$ so y is feasible. Furthermore $-c^\top x + b^\top y \le 0$ so $y^\top b \le c^\top x$ and by Weak Duality we must then have $y^\top b = c^\top x$ and we are finished. Otherwise, by Corollary 3.2 there exists $[u^\top, y^\top, \lambda, w^\top, x^\top] \ge 0$ satisfying

$$[u^\top, y^\top, \lambda, w^\top, x^\top] \begin{bmatrix} -I & 0 \\ A & 0 \\ -c^\top & b^\top \\ 0 & -I \\ 0 & -A^\top \end{bmatrix} = 0 \qquad \text{and} \qquad [u^\top, y^\top, \lambda, w^\top, x^\top] \begin{bmatrix} 0 \\ b \\ 0 \\ 0 \\ -c \end{bmatrix} < 0$$

This gives the following:

$$y^\top A - \lambda c^\top \ge 0 \tag{1}$$

$$Ax - \lambda b \le 0 \tag{2}$$

$$y^\top b < c^\top x \tag{3}$$

If $\lambda > 0$, then scaling the vector $[u^\top, y^\top, \lambda, w^\top, x^\top]$ by $1/\lambda$ we may assume that $\lambda = 1$. However, then (1) and (2) show that x and y are feasible in the primal and dual (respectively) and (3) contradicts Weak Duality.

Otherwise we have $\lambda = 0$. Now, by (3), either $y^\top b < 0$ or $c^\top x > 0$. In the former case, we claim that the dual problem is unbounded (which contradicts the assumption that the primal is feasible). To see this, let y_f be any feasible point in the dual, let μ be a positive number, and consider the vector $y_f + \mu y$. We have $y_f + \mu y \ge 0$ and

$$(y_f + \mu y)^\top A = y_f^\top A + \mu y^\top A \ge c^\top.$$

Thus $y_f + \mu y$ is feasible in the dual and $(y_f + \mu y)^\top b = y_f^\top b + \mu(y^\top b)$ can be made arbitrarily small by choosing μ sufficiently large. If $c^\top x > 0$ then a similar argument shows that the primal is unbounded (again giving us a contradiction). $\square$

Integer Programming and Linear Systems

Integral, Rational: A vector or matrix is *integral* (*rational*) if every entry is integral (*rational*).

Integer Programming: An *integer program* is a linear program with the added constraint that the feasible point must be integral. For instance, the following:

$$\begin{aligned}&\text{maximize } c^\top x\\&\text{s.t. } Ax \le b\\&x \ge 0\\&x \text{ integral}\end{aligned}$$

Unlike linear programming problems which (under suitable assumptions) can be computed in polynomial time, integer programming is known to be NP-hard even in many special cases (so assuming $P \neq NP$, there would be no polynomial time solution). Many classical combinatorial problems can be expressed as integer programs. For instance, if G is a graph and A is the vertex-independent set incidence matrix, then

$$\chi(G) = \min\{1^\top x : Ax \ge 1,\ x \ge 0, \text{ and } x \text{ is integral}\}$$
$$\omega(G) = \max\{y^\top 1 : y^\top A \le 1,\ y \ge 0, \text{ and } y \text{ is integral}\}.$$

Linear System: A *linear system* is any set of linear inequalities. For instance:

$$\begin{aligned}Ax &\le b\\x &\ge 0\end{aligned}$$

We view a linear system as a linear program without the objective function. Now, let us consider extending the above linear system to a linear program using an integral objective function w. Now we define the following parameters:

$$g_w = \max\{w^\top x : Ax \le b,\ x \ge 0 \text{ and } x \text{ integral}\}$$
$$f_w = \min\{y^\top b : y^\top A \ge w,\ y \ge 0 \text{ and } y \text{ integral}\}$$

Dropping the integrality constraints give us the following fractional parameters:

$$g_w^* = \max\{w^\top x : Ax \le b \text{ and } x \ge 0\}$$
$$f_w^* = \min\{y^\top b : y^\top A \ge w \text{ and } y \ge 0\}$$

Now, assuming both the primal and dual are feasible, LP-duality gives us the following:

$$g_w \leq g_w^* = f_w^* \leq f_w \tag{4}$$

Integral: Our linear system is *integral* if $g_w = g_w^*$ for every integral w. Note that our system is integral if and only if the polyhedron $\{x : Ax \leq b, x \geq 0\}$ is integral.

Totally Dual Integral: Our linear system is *totally dual integral* (henceforth abbreviated *TDI*) if $f_w = f_w^*$ for every integral w. Note: here we permit that the dual program to be unbounded.

Theorem 3.5 (Edmonds-Giles) *Let $b \in \mathbb{Z}^S$ and let A be a rational matrix indexed by $R \times S$. If the linear system $Ax \leq b$ and $x \geq 0$ is TDI, then it is integral.*

Proof: Let $P = \{x \in \mathbb{R}^S_+ : Ax \leq b\}$. Then it suffices to show that P is integral. Since P is contained in the nonnegative orthant, it is pointed, so it suffices to show that every vertex of P is an integral point. Let $\tilde{x}$ be a vertex of P and choose an integral objective function $c \in \mathbb{R}^S$ so that $\tilde{x}$ is the unique optimum for the linear program $\max\{c^\top x : x \in P\}$. Now, since c is integral, and our linear system is TDI, we have

$$c^\top \tilde{x} = \max\{c^\top x : \ x \in \mathbb{R}^S_+ \text{ and } Ax \leq b \ \} = \min\{b^\top y : \ y \in \mathbb{R}^R_+ \text{ and } y^\top A \geq c \ \} \in \mathbb{Z}.$$

Choose a positive integer N so that $\tilde{x}$ is still the unique optimum of the linear program $\max\{d^\top x : x \in P\}$ whenever $dist(c, d) < \frac{1}{N}$. Let c_1 be the objective function Nc and note that $c^\top \tilde{x} = Nc^\top \tilde{x}$ is in $\mathbb{Z}$. Next, we let $s \in S$ be given, and we let c_2 be the vector obtained from c_1 by incrementing the coordinate indexed by s by 1. Now, as above, we have

$$c_2^\top \tilde{x} = \max\{c_2^\top x : \ x \in \mathbb{R}^S_+ \text{ and } Ax \leq b \ \} = \min\{b^\top y : \ y \in \mathbb{R}^R_+ \text{ and } y^\top A \geq c_2 \ \} \in \mathbb{Z}.$$

But then, $\tilde{x}(s) = c_2^\top \tilde{x} - c_1^\top \tilde{x} \in \mathbb{Z}$, so the s coordinate of $\tilde{x}$ is an integer. Since s was arbitrary, $\tilde{x}$ is integral, and since $\tilde{x}$ was arbitrary, P is integral. $\square$

Note: In terms of which inequalities in equation (4) hold for all integral w, there are only two interesting possibilities. If $g_w = g_w^*$ for every integral w, then our system is integral. If $f_w = f_w^*$ holds for all integral w, then our system is TDI, but then the above theorem tells us that $g_w = g_w^*$ also holds for all w, so we have $f_w = g_w$ for all integral w.

Packing and Covering Systems

Packing and Covering Systems: Let $\mathcal{C}$ be a clutter with clutter matrix M. We define the following two linear systems for $\mathcal{C}$.

Packing System	*Covering System*
$Mx \geq 1$	$Mx \leq 1$
$x \geq 0$	$x \geq 0$

Note that the set of feasible points for the packing system is precisely our packing polyhedron while the feasible points for the covering system constitute our covering polyhedron.

Objective Function 1: It is natural to extend the above systems to linear programs by using the objective function 1. Considering the integer programs associated with these LP's and their duals, we rediscover our packing and covering parameters as follows:

$$\tau(\mathcal{C}) = \min\{1^\top x : Mx \geq 1,\, x \geq 0, \text{ and } x \text{ integral}\}$$
$$\nu(\mathcal{C}) = \max\{y^\top 1 : y^\top M \leq 1,\, y \geq 0, \text{ and } y \text{ integral}\}$$
$$\alpha(\mathcal{C}) = \max\{1^\top x : Mx \leq 1,\, x \geq 0, \text{ and } x \text{ integral}\}$$
$$\kappa(\mathcal{C}) = \min\{y^\top x : y^\top M \geq 1,\, y \geq 0, \text{ and } y \text{ integral}\}$$

Fractional Parameters: Dropping the integrality constraints, gives us the following fractional parameters:

$$\tau^*(\mathcal{C}) = \min\{1^\top x : Mx \geq 1,\, x \geq 0\}$$
$$\nu^*(\mathcal{C}) = \max\{y^\top 1 : \; y^\top M \leq 1,\, y \geq 0\}$$
$$\alpha^*(\mathcal{C}) = \max\{1^\top x : \; Mx \leq 1,\, x \geq 0\}$$
$$\kappa^*(\mathcal{C}) = \min\{y^\top 1 : \; y^\top M \geq 1,\, y \geq 0\}$$

Note that by LP duality, we have the following two equations:

$$\nu(\mathcal{C}) \leq \nu^*(\mathcal{C}) = \tau^*(\mathcal{C}) \leq \tau(\mathcal{C}) \tag{5}$$
$$\kappa(\mathcal{C}) \geq \kappa^*(\mathcal{C}) = \alpha^*(\mathcal{C}) \geq \alpha(\mathcal{C}) \tag{6}$$

Also note that $\mathcal{C}$ packs if we have equalities in (5) and $\mathcal{C}$ covers if we have equalities in (6).

Example: Let G be a graph and let $\mathcal{C} = IN(G)$ (the independence-node clutter of G). Then we have

$$\omega(G) = \alpha(\mathcal{C}) \leq \alpha^*(\mathcal{C}) = \kappa^*(\mathcal{C}) \leq \kappa(\mathcal{C}) = \chi(G)$$

Here, the parameters $\alpha^*(\mathcal{C})$ and $\kappa^*(\mathcal{C})$ are usually called the fractional clique number of G and the fractional chromatic number of G.

General Objective Functions: For $w \in \mathbb{Z}_+^V$, we may extend the packing or covering system to a linear program by using w as an objective function. As with $w = 1$, we rediscover our weighted packing and covering parameters as follows.

$$\begin{aligned}
\tau_w &= \min\{w^\top x : x \in \mathbb{Z}_+^V \text{ and } Mx \geq 1\} \\
\nu_w &= \max\{y^\top 1 : y \in \mathbb{Z}_+^E \text{ and } y^\top M \leq w\} \\
\alpha_w &= \max\{w^\top x : x \in \mathbb{Z}_+^V \text{ and } Mx \leq 1\} \\
\kappa_w &= \min\{y^\top 1 : y \in \mathbb{Z}_+^E \text{ and } y^\top M \geq w\}
\end{aligned}$$

As before, we shall drop these integrality constraints to define fractional parameters:

$$\begin{aligned}
\tau_w^* &= \min\{w^\top x : x \in \mathbb{R}_+^V \text{ and } Mx \geq 1\} \\
\nu_w^* &= \max\{y^\top 1 : y \in \mathbb{R}_+^E \text{ and } y^\top M \leq w\} \\
\alpha_w^* &= \max\{w^\top x : x \in \mathbb{R}_+^V \text{ and } Mx \leq 1\} \\
\kappa_w^* &= \min\{y^\top 1 : y \in \mathbb{R}_+^E \text{ and } y^\top M \geq w\}
\end{aligned}$$

General Packing & Covering Inequalities:

$$\nu_w \leq \nu_w^* = \tau_w^* \leq \tau_w$$
$$\kappa_w \geq \kappa_w^* = \alpha_w^* \geq \alpha_w$$

Proposition 3.6 *For every clutter $\mathcal{C}$, we have that $\mathcal{C}$ satisfies*

Ideal	$\Leftrightarrow$	$\tau_w = \tau_w^*$ *for all* $w \in \mathbb{Z}_+^V$	$\Leftrightarrow$	*the packing system is integral*
MFMC	$\Leftrightarrow$	$\tau_w = \nu_w$ *for all* $w \in \mathbb{Z}_+^V$	$\Leftrightarrow$	*the packing system is TDI*
Perfect	$\Leftrightarrow$	$\alpha_w = \alpha_w^*$ *for all* $w \in \mathbb{Z}_+^V$	$\Leftrightarrow$	*the covering system is integral*
Perfect$^+$	$\Leftrightarrow$	$\alpha_w = \kappa_w$ *for all* $w \in \mathbb{Z}_+^V$	$\Leftrightarrow$	*the covering system is TDI*

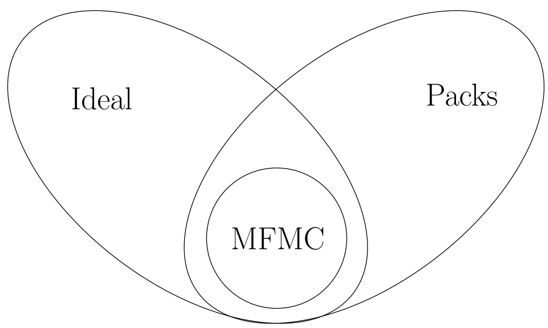
Proof: It follows immediately from the definition that $\mathcal{C}$ is ideal if and only if the packing system is integral and this implies that $\tau_w = \tau_w^*$ for all $w \in \mathbb{Z}^V$ (not only for $w \geq 0$).

However, for every vertex x of the packing polyhedron, there exists $w \in \mathbb{Z}_+^V$ and $\lambda \geq 0$ so that $\{x\} = \{y \in Pack(\mathcal{C}) : w^\top y \leq \lambda\}$ and it follows from this that $\tau_w = \tau_w^*$ for every $w \in \mathbb{Z}_+^V$ implies that the packing polyhedron is integral (and thus that $\mathcal{C}$ is ideal). A similar argument yields the equivalences involving perfect.

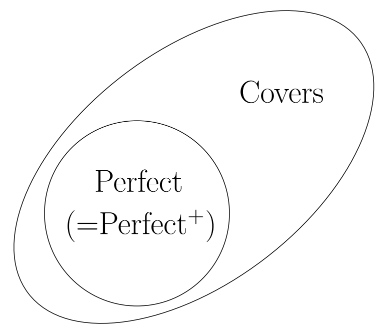
We have by definition that $\mathcal{C}$ is MFMC if and only if $\tau_w = \nu_w$ for all $w \in \mathbb{Z}_+^V$. Let $w \in \mathbb{Z}^V$ and consider the linear programs given by ν_w^* and τ_w^*. We find that either $w \geq 0$ and both programs are feasible, or $w \ngeq 0$ and ν_w^* is infeasible and τ_w^* is unbounded. The remaining equivalence follows easily from this. The proof of the equivalences involving Perfect$^+$ is somewhat similar. $\square$

Corollary 3.7 *For a clutter $\mathcal{C}$, we have MFMC $\Rightarrow$ Ideal, and Perfect$^+$ $\Rightarrow$ Perfect.*

Note: We shall prove later that Perfect $\Leftrightarrow$ Perfect$^+$, giving us the following Venn diagrams.



Covering



Packing