4 Packing T-joins and T-cuts

Introduction

Graft: A *graft* consists of a connected graph $G = (V, E)$ with a distinguished subset $T \subseteq V$ where $|T|$ is even.

T-cut: A *T-cut* of G is an edge-cut C which separates T into two sets of odd size. We let $TC(G)$ denote the clutter of all minimal T-cuts.

T-join: A *T-join* of G is a subset $J \subseteq E$ with the property that $\{v \in V : v \text{ has odd degree in } (V, J)\} = T$. We let $TJ(G)$ denote the clutter of all minimal T-joins.

Observation 4.1 *If $C \subseteq E$ is a T-cut and $J \subseteq E$ is a T-join, then $C \cap J \neq \emptyset$.*

Proof: Suppose (for a contradiction) that $C \cap J = \emptyset$ and let $\{X, Y\}$ be the partition of V resulting from the edge cut C. Now, consider the subgraph H of (V, J) induced by X. The vertices of odd degree in H are precisely $X \cap T$, but this set has odd size, which is contradictory. $\square$

Observation 4.2

(i) *A T-cut $C = \delta(X)$ is a minimal T-cut if and only if the graphs induced on X and $V \setminus X$ are connected.*

(ii) *A T-join $J \subseteq E$ is a minimal T-join if and only if the graph (V, J) is a forest.*

Proof: Homework.

Proposition 4.3 *$TJ(G)$ and $TC(G)$ are blocking clutters for every graft G.*

Proof: Homework.

Examples

1. If $T = \{s, t\}$, then the T-cuts are precisely the st-cuts, and the minimal T-joins are precisely the st-paths (check!).

2. If $T = V$, then the T-joins of size $\frac{1}{2}|V|$ are precisely the perfect matchings of G. So, for instance, if G is r-regular, then G is r-edge-colourable if and only if $\nu(TJ(G)) = r$ (check!).

Ideal for T-joins and T-cuts

Constraints: Let A be a matrix indexed by $R \times S$, let $b \in \mathbb{R}^R$ and let $P = \{x \in \mathbb{R}^S : Ax \leq b\}$. Every $r \in R$ places the *constraint* on P that every point in P must have dot product with row r of A at most b_r. A point $x \in P$ is *tight* with respect to r if we have equality. We say that r *uses* $s \in S$ if $A_{rs} \neq 0$.

Proposition 4.4 *If x is a vertex of the polyhedron P, then the following hold.*

(i) *If L is a line through x, then $P \cap L$ is a closed (possibly one-way infinite) interval of L with endpoint x.*

(ii) *If $S' \subseteq S$, then there must be at least $|S'|$ constraints which are tight with respect to x and use an element in S'.*

Proof: Part (i) is an immediate consequence of the definition of vertex. For part (ii), suppose (for a contradiction) that $R' \subseteq R$ is the set of tight constraints which use some $s \in S'$ and that $|R'| < |S'|$. Then we may choose a nonzero vector $y \in \mathbb{R}^S$ supported on S' which is orthogonal to the row of A indexed by r for every $r \in R'$. Now the line through x in the direction of y contradicts (i). □

Lemma 4.5 *Let $\tilde{x}$ be a vertex of the polyhedron*

$$P' = \{x \in \mathbb{R}^E_+ : x(\delta(v)) \geq 1 \text{ for every } v \in T\}.$$

Let $\tilde{G}$ be the subgraph of G induced by the edges $e \in E$ for which $\tilde{x}(e) > 0$. Then every component of $\tilde{G}$ is one of the following.

(i) *an odd cycle with all vertices in T and all edges have $\tilde{x}(e) = \frac{1}{2}$*

(ii) *a star with all leaf vertices in T and all edges have $\tilde{x}(e) = 1$.*

Proof: Note that the constraints for the polyhedron P' consist of $x(e) \geq 0$ for every $e \in E$ and $x(\delta(v)) \geq 1$ for every $v \in T$, so we index these constraints by $E \cup T$. Let H be a component of $\tilde{G}$, let Y be the vertex set of H, and let $S \subseteq E$ be the set of edges with both ends in Y. By applying (ii) of the previous proposition to S, we have

$$|\{e \in S : \tilde{x}(e) = 0\}| + |\{v \in Y \cap T : \tilde{x}(\delta(v)) = 1\}| \geq |S|. \tag{1}$$

The next inequality follows by subtracting $|\{e \in S : \tilde{x}(e) = 0\}|$ from both sides of (1).

$$|V(H)| \geq |\{v \in Y \cap T : \tilde{x}(\delta(v)) = 1| \geq |\{e \in S : \tilde{x}(e) > 0\}| = |E(H)|. \tag{2}$$

It follows from (2) that H has at most one cycle.

First consider the case that H contains a cycle. Then all inequalities in (2) must be equalities, so every vertex $v \in V(H)$ is in T and satisfies $\tilde{x}(\delta(v)) = 1$. If H consists of more than just a cycle, then H must have a leaf vertex v with leaf edge uv. Then we must have $\tilde{x}(uv) = 1$ to satisfy the tight constraint at v, but then u cannot be incident with another edge with $\tilde{x}(e) > 0$ - a contradiction. Thus, H must be a cycle. If H is an even cycle, let $y \in \mathbb{R}^E$ be the vector which is alternately $+1$ and -1 on edges of H, and 0 on all other edges. Now, for any $\epsilon < \min_{e \in E(H)} \tilde{x}(e)$, the point $\tilde{x} + \epsilon y$ is in P', but this contradicts (i) of the previous proposition. Thus, H must be an odd cycle. Now, if the edge $e \in E(H)$ has $\tilde{x}(e) = \alpha$ and f is an adjacent edge of H, then $\tilde{x}(f) = 1 - \alpha$. It follows easily from this that every edge e of H must have $\tilde{x}(e) = \frac{1}{2}$, thus (i) holds.

Next suppose that H is a tree. Now, by (2), all but one vertex of H is a vertex in T for which $\tilde{x}(\delta(v)) = 1$. In particular, there must be a leaf vertex v with leaf edge uv for which $v \in T$ and $\tilde{x}(\delta(v)) = 1$. Then $\tilde{x}(uv) = 1$ and either u is also a leaf vertex of H, so H is a one edge graph and we are finished, or u is not a leaf vertex and is the unique vertex of H for which the star constraint is not tight. In this latter case, it follows from the previous argument that every leaf vertex of H is a vertex of T which is adjacent to u and has leaf edge e with $\tilde{x}(e) = 1$. Thus (ii) holds. $\square$

Theorem 4.6 (Edmonds-Johnson) *The clutter $TC(G)$ is ideal.*

Proof: Let $\mathcal{C} = TC(G)$ and define the polyhedra

$$P = Pack(\mathcal{C})$$
$$P_I = Up(b(\mathcal{C}))$$

So, P is the set of edge weights which give every T-cut a total weight of ≥ 1 and P_I is the up-hull of the set of incidence vectors of minimal T-joins. Proving that $\mathcal{C}$ is ideal is equivalent to showing that $P = P_I$, or equivalently that P is an integral polyhedron. This we shall prove by induction on $|V|$. Note that by construction, we have $P_I \subseteq P \subseteq P'$ where P' is the polyhedron from Lemma 4.5. Next, let $\tilde{x}$ be a vertex of P and let $\tilde{G}$ be the subgraph of G induced by the edges e with $\tilde{x}(e) > 0$.

First, suppose that $\tilde{x}$ is also a vertex of P' and consider a component H of the graph $\tilde{G}$ with $Y = V(H)$. Since $\tilde{x}(\delta(Y)) = 0$, it must be that $|Y \cap T|$ is even (otherwise $\delta(Y)$ contains a T-cut, so $\tilde{x}$ must give it weight ≥ 1). It now follows from the previous lemma that either H is a star with $\tilde{x}(e) = 1$ for every edge in H, and either H has an odd number of leaves and center vertex in T or H has an even number of leaves and center not in T. Since this must hold for every component of $\tilde{G}$, we find that $\tilde{x}$ is the incidence vector of a T-join, so in particular, $\tilde{x}$ is integral.

Next, suppose that $\tilde{x}$ is not a vertex of P'. In this case, there must be a constraint of the polyhedron P which is tight for $\tilde{x}$ but is not a constraint of the polyhedron P'. That is, there must be a T-cut $C \subseteq E$ so that $\tilde{x}(C) = 1$ and so that $C \neq \delta(v)$ for any $v \in V$. Let C partition the vertices into Y_1 and Y_2, and for $i = 1, 2$ form a new graft G_i from G by identifying Y_i to a single new vertex, y_i and declaring this vertex to be in the distinguished subset. Let $\tilde{x}_i$ be the edge weighting on G_i induced by $\tilde{x}$. Since every T-cut of G_i is also a T-cut of G we find (by induction) that $\tilde{x}$ may be written as a convex combination of incidence vectors of T-joins in G_i. Since each of these T-joins must use exactly one edge of C they may be combined to give us a convex combination of incidence vectors of T-cuts of G which sum to $\tilde{x}$. This proves that $\tilde{x}$ lies in P_I which completes our proof. □

Corollary 4.7 *The clutter $TJ(G)$ is ideal.*

Proof: Since $TJ(G)$ and $TC(G)$ are blocking clutters, this follows immediately from Lehman's theorem. □

MFMC for T-joins

Packing Parameters: We will focus on the clutter of T-joins in this section, so for a graft G, we let $\nu(G) = \nu(TJ(G))$ and $\tau(G) = \tau(TJ(G))$ (and similarly for ν_w, τ_w). As usual, when the graft is clear from context, we simplify the notation to ν and τ.

Odd $\mathbf{K_{2,3}}$: Any graft isomorphic to the one depicted below is called an *odd* $K_{2,3}$ (here filled in nodes are in T and empty ones are not).

Note that the clutter of T-joins of this graft is isomorphic to Q_6, so in particular, it has $\nu = 1 < 2 = \tau$.

Graft Minors: Let $G = (V, E)$ be a connected graft with distinguished subset $T \subseteq V$. To *delete* an edge e or a vertex $v \in V \setminus T$ we simply delete this from the graph. To *contract* an edge $e = uv$ we contract the edge in the graph to form a new vertex, say w and then we modify T by removing u, v and then adding w if and only if $|T \cap \{u, v\}| = 1$. Any graft obtained from G by a sequence of such deletions and contractions is called a *minor* of G. It will be helpful at times to contract larger subgraphs; if $H \subseteq G$ is connected then to *contract* H we choose a spanning tree F of H, delete all edges in $E(H) \setminus E(F)$ and then contract the edges in F (Note that the resulting graft does not depend on the chosen tree).

Lemma 4.8 *If $G = (V, E)$ is a minor minimal graft with $\nu(G) < \tau(G)$ then G is an odd $K_{2,3}$.*

Proof: We shall establish the proof in steps. For any $F \subseteq E$ we let $odd(F) = \{v \in V : v \text{ has odd degree in } (V, F)\}$. We set $Y = \{v \in T : deg(v) = \tau\}$.

(0) $T \Delta odd(F)$ is even for every $F \subseteq E$.

Since $odd(F)$ is the set of vertices of odd degree in the graph (V, F) it must have even size. Since T also has even size, it follows that $T \Delta odd(F)$ is even.

(1) For every $e \in E$ we have $\tau - 1 \geq \nu \geq \nu(G \setminus e) = \tau(G \setminus e) \geq \tau - 1$.

The only nontrivial relation above is $\nu(G \setminus e) = \tau(G \setminus e)$ and this follows from the assumption that G is minor minimal with $\nu < \tau$.

(2) $\nu = \tau - 1$

This is an immediate consequence of (1).

(3) Every edge is in a minimum size T-cut.

Were $e \in E$ not contained in a minimum size T-cut, we would have $\tau(G \setminus e) = \tau$ which contradicts (1).

(4) $|V \setminus Y| \geq 2$.

Choose edge-disjoint T-joins $J_1, J_2, \ldots, J_\nu$ and set $F = E \setminus (\cup_{i=1}^{\tau} J_i)$. Now, consider a vertex $v \in Y$. Since $deg(v) = \tau = \nu + 1$ and we have ν disjoint T-joins each of which contains an odd number of edges in $\delta(v)$, it must be that each J_i contains exactly one edge of $\delta(v)$, so F also contains one edge of $\delta(v)$. In particular $Y \subseteq odd(F)$. If $odd(F) = T$, then F is a T-join, giving us the contradiction $\nu = \tau$. Otherwise $odd(F) \Delta T$ is even, and this implies (4).

(5) Every minimum size T-cut is of the form $\delta(v)$ for some $v \in Y$.

Suppose (for a contradiction) that there is a T-cut C of size τ which partitions V into $\{V_1, V_2\}$ where $|V_1|, |V_2| \geq 2$. Now, for $i = 1, 2$ we form a new graft G_i with distinguished vertex set T_i by identifying V_i to a single new vertex v_i (deleting any newly created loops), and setting T_i to be $(T \cap V(G_i)) \cup \{v_i\}$. Now, G_i has the T-cut $\delta(v_i)$ of size τ. Furthermore, every T-cut of G_i is also a T-cut of G (to see this, blow up the vertex v_i to V_i and return to the original graft). Thus, we must have $\tau(G_i) = \tau$. Since G_i is a proper minor of G we have $\nu(G_i) = \tau(G_i) = \tau$, so we may choose τ disjoint T-joins $J_1^i, \ldots, J_\tau^i$ of G_i for $i = 1, 2$. Since C is a T-cut of size τ in both G_1 and G_2, every T-join J_k^i must contain exactly one edge of C, so we may assume by reordering that $J_k^1 \cap C = J_k^2 \cap C$ for every k. Now $J_1^1 \cup J_1^2, \ldots J_\tau^1 \cup J_\tau^2$ is a list of τ disjoint T-joins in G, giving us a contradiction and completing the proof of (5).

(6) G is 2-connected.

If G is not connected, then by minimality, every component of G has ν disjoint T-joins, so G does as well. Thus, G must be connected. If G is not 2-connected, then we may choose two nontrivial subgraphs $G_1, G_2 \subseteq G$ so that $\{E(G_1), E(G_2)\}$ is a partition of $E(G)$ and $V(G_1) \cap V(G_2) = \{v\}$. For $i = 1, 2$ we extend G_i to a graft by declaring the distinguished subset of vertices to be $(T \cap (V(G_i)) \setminus \{v\}$ if this set is even, and $(T \cap V(G_i)) \cup \{v\}$ otherwise. It now follows that every T-cut of G_i is also a T-cut of G, so $\nu(G_i) \geq \nu$ and by minimality, we may choose T-joins $J_1^1, J_2^1, \ldots, J_\nu^1$ of G_1 and $J_1^2, J_2^2, \ldots, J_\nu^2$ of G_2. Now $J_1^1 \cup J_1^2, J_2^1 \cup J_2^2, \ldots, J_\nu^1 \cup J_\nu^2$ is a list of ν disjoint T-joins in G, giving us a contradiction.

(7) There does not exist a 2 vertex cut $\{u, v\}$ with $u \in Y$.

Suppose (for a contradiction) that (7) is false and choose subgraphs $H_1, H_2 \subseteq G \setminus u$ so that $E(H_1) \cup E(H_2) = E(G \setminus u)$ and so that $V(H_1) \cap V(H_2) = \{v\}$. Now for $i = 1, 2$ let G_i be the graft obtained from G by T-contracting G_i. Since every edge-cut of G_i is also an edge-cut of G we have that $\tau(G_i) \geq \tau$ so by induction we may choose τ disjoint T-joins $F_1^i, F_2^i, \ldots, F_\tau^i$ of G_i for $i = 1, 2$. Since the vertex $u \in Y$ every F_j^i contains exactly one edge of $\delta(u)$ and by reordering, we may assume that $F_j^1 \cap \delta(u) = F_j^2 \cap \delta(u)$ for every $1 \leq j \leq \tau$. Now let $F_j = F_j^1 \cup F_j^2$ for $1 \leq j \leq \tau$. For every vertex $w \in V(G) \setminus \{v\}$ we have that $w \in odd(F_j)$ if and only if $w \in T$. But then it follows from (0) that $odd(F_j) = T$. Thus we have found τ disjoint T-joins giving us a contradiction.

With (0)-(7) we are now ready to complete the proof. By (4) we may choose two vertices $x_1, x_2 \in V(G) \setminus Y$ of minimum distance (note that x_1, x_2 cannot be adjacent by (3) and (5)). If there do not exist three internally disjoint paths from x_1 to x_2, then there is a one or two vertex separation of the graph with x_1 on one side and x_2 on the other. But then (7) shows that every vertex in this separation is in $V(G) \setminus Y$ and then a shortest path from x_1 to x_2 must contain another vertex of $V(G) \setminus Y$ which contradicts our choice of x_1, x_2. It follows from this that we may choose three internally disjoint paths, say $P_{1,2}$, P_3 from x_1 to x_2. For $i = 1, 2, 3$ let y_i be the neighbour of x_1 on P_i and note that $y_i \in Y$ by (3) and (5). We now split into two cases:

Case 1: $G \setminus \{y_1, y_2, y_3\}$ is not connected.

Let $H_1, H_2, \ldots, H_k$ be the components of $G \setminus \{y_1, y_2, y_3\}$ and assume that $H_1, \ldots, H_j$ are the components which contain an odd number of vertices in T. Note that by parity we must have j odd. First suppose that $j \geq 3$. Then every $\delta(H_i)$ for $1 \leq i \leq j$ is a T-cut, these T-cuts are all disjoint, and their union is contained in $\delta(y_1) \cup \delta(y_2) \cup \delta(y_3)$. This is only possible if $j = 3$ and each of $\delta(H_i)$ for $1 \leq i \leq 3$ is a minimum size T-cut and $\cup_{i=1}^3 \delta(H_i) = \cup_{i=1}^3 \delta(y_i)$. But then by (5) we must have that each of H_1, H_2, H_3 contains just a single point in Y and since there is a vertex of Y somewhere in the graph we must have $k > 3$, but then we have a contradiction to connectivity. Thus we must have $j = 1$ and (since $\{y_1, y_2, y_3\}$ is connected) $k \geq 2$. Now contract every H_i to a single point and delete those newly created points for $i \geq 3$. The resulting graft is an odd $K_{2,3}$.

Case 2: $G \setminus \{y_1, y_2, y_3\}$ is connected.

Let H_1 be the subgraph consisting of the single vertex x_1 and let H_2 be the subgraph consisting of $P_1 \cup P_2 \cup P_3 \setminus \{y_1, y_2, y_3, x_1\}$. It follows from the connectivity of $G \setminus \{y_1, y_2, y_3\}$ that we may extend H_1, H_2 so connected subgraphs H_1', H_2' so that $\{V(H_1), V(H_2)\}$ is a partition of $V(G) \setminus \{y_1, y_2, y_3\}$. Now contract H_1 and H_2 to a single vertex. It follows from parity that exactly one of H_1 or H_2 has an odd number of vertices in T, so the resulting graft is isomorphic to an odd $K_{2,3}$. This completes the proof. □

Theorem 4.9 *If $\mathcal{C}$ is a clutter of T-joins, then $\mathcal{C}$ is MFMC if and only if $\mathcal{C}$ has no Q_6 minor.*

Proof: If $\mathcal{C}$ has Q_6 as a minor then assigning each edge which was deleted in this minor creation a weight of 0 and each edge which was contracted a weight of ∞ and each remaining edge a weight of 1 results in a weighting for which $\nu = 1$ and $\tau = 2$. If $\mathcal{C}$ has no Q_6 minor, then choose a graft $G = (V, E)$ so that $\mathcal{C} = TJ(G)$, and let $w \in \mathbb{Z}_+^E$. Next, modify G to form a new graft G' by replacing every edge e with $w(e)$ copies of e. Now, G' has no odd $K_{2,3}$ minor (otherwise G would have an odd $K_{2,3}$ minor), so by the lemma we have $\nu_w(G) = \nu(G') = \tau(G') = \tau_w(G)$. Since w was arbitrary, $\mathcal{C}$ has the MFMC property. □