3 The Goemans-Williamson Algorithm

We let $S^k = \{x \in \mathbb{R}^{k+1} : ||x|| = 1\}$. Throughout we fix a graph $G = (V, E)$ and set $n = |V|$.

Maxcut: We let $\text{Maxcut}(G)$ denote the size of the largest edge cut in G.

Theorem 3.1

(i) *It is NP-complete to decide if G satisfies* $\text{Maxcut}(G) \geq k$.

(ii) *It is NP-hard to approximate* $\text{Maxcut}(G)$ *to within a factor of* $\frac{16}{17}$ *unless* $P = NP$.

(iii) *Assuming* $BPP \neq NP$ *and the Unique Games Conjecture, there is no approximation algorithm for* $\text{Maxcut}(G)$ *achieving a better ratio than Goemans-Williamson algorithm.*

Semidefinite Relaxation: First, observe that

$$\text{Maxcut}(G) = \max \frac{1}{2} \sum_{ij \in E} (1 - x_i \cdot x_j)$$
$$\text{s.t. } x_i \in \{-1, 1\} \text{ for every } i \in V$$

Noting that $\{-1, 1\} = S^0$, we consider the following semidefinite relaxation of this problem

$$\text{Maxcut}^\circ(G) = \max \frac{1}{2} \sum_{ij \in E} (1 - x_i \cdot x_j)$$
$$\text{s.t. } x_i \in S^{n-1} \text{ for every } i \in V$$

Note that $\text{Maxcut}(G) \leq \text{Maxcut}^\circ(G)$. Also, since $||x_i|| = 1$ for every $i \in V$ we have

$$\frac{1}{4} \sum_{ij \in E} ||x_i - x_j||^2 = \frac{1}{4} \sum_{ij \in E} (x_i - x_j) \cdot (x_i - x_j) = \frac{1}{2} \sum_{ij \in E} (1 - x_i \cdot x_j)$$

So, in other words, the problem $\text{Maxcut}^\circ(G)$ is equivalent to embedding the vertices on the sphere S^{n-1} so that the sum of the squares of the corresponding edge lengths is maximum.

Gram matrix: Given a collection of vectors $\{x_i\}_{i \in S} \in \mathbb{R}^m$ the associated *Gram matrix* $X \in \mathbb{R}^{S \times S}$ is given by $X_{ij} = x_i \cdot x_j$. If $S = \{1, \ldots, n\}$ and we form a matrix U by taking x_i as the i^{th} column, then $X = U^\top U$. Note that X is a gram matrix if and only if $X \succeq 0$.

Theorem 3.2 *Let A be the adjacency matrix of G and let X be the gram matrix of $\{x_i\}_{i\in V} \in \mathbb{R}^n$. Then $\{x_i\}_{i\in V}$ is optimal for* Maxcut°(G) *if and only if X is optimal for the SDP*

$$\begin{aligned}&\min X \cdot A\\ &\textit{s.t. } X \succeq 0\\ &X_{ii} = 1 \textit{ for every } i \in V\end{aligned}$$

Proof: It is immediate from the definition that $X \succeq 0$. Therefore,

$$\begin{aligned}X \text{ is feasible } &\Leftrightarrow X_{ii} = 1 \text{ for every } i \in V\\ &\Leftrightarrow ||x_i|| = 1 \text{ for every } i \in V\\ &\Leftrightarrow \{x_i\}_{i\in V} \text{ feasible for Maxcut}^\circ(G).\end{aligned}$$

Furthermore,

$$\frac{1}{2}\sum_{ij\in E}(1 - x_i \cdot x_j) = \frac{1}{2}|E| - \frac{1}{4}A \cdot X$$

The result follows immediately from this. □

Note: It follows from the above that an (approximately) optimal solution for Maxcut°(G) can be computed in polynomial time.

Theorem 3.3 *There exists $\alpha \in \mathbb{R}$ with $\alpha \sim .868$ with the following property. If $\{x_i\}_{i\in V}$ is optimal for* Maxcut°, *H is a random hyperplane through the origin, and $Cut(H)$ is the size of the edge cut consisting of those edges $ij \in E$ for which x_i and x_j are separated by H, then*

$$\mathbb{E}[Cut(H)] \geq \alpha\, \text{Maxcut}^\circ(G) \geq \alpha\, \text{Maxcut}(G)$$

Proof: Let β be the largest real number with the property that $\arccos(t) \geq \beta(1-t)$ for every $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and set $\alpha = \frac{2\beta}{\pi}$. The probability that an edge $ij \in E$ will have ends on either side of this partition is precisely $\frac{\theta}{\pi}$ where θ is the angle between the vectors x_i and x_j. Thus

$$\begin{aligned}\mathbb{E}[Cut(H)] &= \sum_{ij\in E}\frac{\arccos(x_i \cdot x_j)}{\pi}\\ &\geq \sum_{ij\in E}\frac{\beta}{\pi}(1 - x_i \cdot x_j)\\ &= \alpha\, \text{Maxcut}^\circ(G)\end{aligned}$$

Goemans-Williamson Algorithm: Compute an (approximately) optimal solution to Maxcut$^\circ(G)$, take a random hyperplane through the origin, and return the corresponding edge cut. By the theorem, the expected size of this edge cut is at least .868 of Maxcut(G) (i.e. this is a .868 approximation algorithm).