5 Ellipsoids

**Ellipsoid:** Recall that an ellipsoid is a set of the form

\[ P = \{ x \in \mathbb{R}^n : (x - a)^\top A(x - a) \leq 1 \} \tag{1} \]

where \( A \) is a (positive) definite matrix and \( a \in \mathbb{R}^n \). Here the point \( a \) is called the center of the ellipsoid. Note that the unit ball \( B_n = \{ x \in \mathbb{R}^n : ||x|| \leq 1 \} \) is an ellipsoid.

**Proposition 5.1**

(i) A set \( P \subseteq \mathbb{R}^n \) is an ellipsoid if and only if \( P \) is the image of \( B_n \) under an affine transformation.

(ii) If \( P \) is an ellipsoid given by \( A, a \) as in (1), then \( \text{Vol}(P) = (\det(A))^{-\frac{1}{2}} \)

**Proof:** Let \( P \) be given by \( A, a \) as in (1). Since \( A \) is positive definite, we may write \( A = U^\top U \) where \( U \) is invertible. Now, consider applying the affine transformation \( x \rightarrow U(x - a) \). This maps \( P \) to the set

\[ U(P) = \{ U(x - a) : (x - a)^\top U^\top U(x - a) \leq 1 \} = \{ y : y^\top y \leq 1 \} = B_n \]

(here \( B_n \) is the \( n \)-dimensional ball of radius 1). So, this mapping sends \( P \) to \( B_n \). The proof of (i) follows immediately from this. For (ii), note that the volume of \( P \) is

\[ \text{Vol}(P) = \det(U^{-1})\text{Vol}(B_n) = (\det(A))^{-\frac{1}{2}}\text{Vol}(B_n). \]

\[ \square \]

**Theorem 5.2 (Löwner-John)** If \( K \subseteq \mathbb{R}^n \) is compact, then there is a unique ellipsoid \( LJ(K) \) of minimum volume containing \( K \). Furthermore, if \( K \) is convex, then scaling \( LJ(K) \) by a factor of \( \frac{1}{n} \) results in an ellipsoid which is contained in \( K \).

**Lemma 5.3** Let \( P \subseteq \mathbb{R}^n \) be an ellipsoid centered at \( a \) and let \( H \) be a halfspace defined by a hyperplane which contains \( a \). Then there is an ellipsoid \( P' \) with the following properties:

(i) \( P \cap H \subseteq P' \)

(ii) \( \text{Vol}(P') \leq e^{-\frac{1}{2(n+1)}} \text{Vol}(P) \)
Proof: First consider the special case that $P = B_n$. In this case, we may assume (by possibly performing a rotation) that $H = \{ x \in \mathbb{R}^n : x_1 \geq 0 \}$. We claim that

$$P' = \left\{ x \in \mathbb{R}^n : \frac{(n+1)^2}{n^2} \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2 - 1}{n^2} \sum_{i=2}^{n} x_i^2 \leq 1 \right\}$$

has the desired properties. First, note that $P'$ is an ellipsoid with center $(\frac{1}{n+1}, 0, \ldots, 0)$ and matrix $A$ a diagonal matrix with $\frac{(n+1)^2}{n^2}$ in the first position and $\frac{n^2 - 1}{n^2}$ in the others. To prove (ii) we need to compute $\text{Vol}(P')$. Let $U$ be the diagonal matrix with $U^2 = A$. Then we have

$$\frac{\text{Vol}(P')}{\text{Vol}(B_n)} = (\det(A))^{-\frac{1}{2}}$$

$$= \left( \frac{n}{n+1} \right)^{\frac{n-1}{2}} \left( \frac{n}{n^2 - 1} \right)^{\frac{n-1}{2}}$$

$$\leq e^{-\frac{1}{n+1}} e^{\frac{n-1}{2(n+1)^2}} = e^{-\frac{1}{n+1}} e^{\frac{1}{2(n+1)}} = e^{-\frac{1}{2(n+1)}}$$

Here we have used the fact that $1 + x \leq e^x$ for all $x$. Now, to prove (i), let $x \in H \cap P$ and observe that

$$\frac{(n+1)^2}{n^2} \left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2 - 1}{n^2} \sum_{i=2}^{n} x_i^2$$

$$= \frac{2(n+1)}{n^2} x_1^2 - \frac{2(n+1)}{n^2} x_1 + \frac{1}{n^2} + \frac{n^2 - 1}{n^2} \sum_{i=1}^{n} x_i^2$$

$$\leq \frac{2(n+1)}{n^2} x_1 (x_1 - 1) + \frac{1}{n^2} + \frac{n^2 - 1}{n^2}$$

$$\leq 1.$$

This completes the proof in the special case when $P = B_n$. For the general case, choose an affine linear transformation $f(x) = Bx + b$ so that $f$ maps $P$ to $B_n$. Then apply the special case to choose $P'$ and then apply the transformation $f^{-1}$. Since affine transformations preserve volume ratios, this yields a suitable ellipsoid for $P$. \qed