5 Ellipsoids

Ellipsoid: Recall that an ellipsoid is a set of the form

$$P = \{x \in \mathbb{R}^n : (x-a)^\top A(x-a) \leq 1\} \tag{1}$$

where A is a (positive) definite matrix and $a \in \mathbb{R}^n$. Here the point a is called the *center* of the ellipsoid. Note that the unit ball $B_n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ is an ellipsoid.

Proposition 5.1

(i) *A set $P \subseteq \mathbb{R}^n$ is an ellipsoid if and only if P is the image of B_n under an affine transformation.*

(ii) *If P is an ellipsoid given by A, a as in (1), then* $\mathrm{Vol}(P) = (\det(A))^{-\frac{1}{2}}$

Proof: Let P be given by A, a as in (1). Since A is positive definite, we may write $A = U^\top U$ where U is invertible. Now, consider applying the affine transformation $x \to U(x-a)$. This maps P to the set

$$U(P) = \{U(x-a) : (x-a)^\top U^\top U(x-a) \leq 1\} = \{y : y^\top y \leq 1\} = B_n$$

(here B_n is the n-dimensional ball of radius 1). So, this mapping sends P to B_n. The proof of (i) follows immediately from this. For (ii), note that the volume of P is

$$\mathrm{Vol}(P) = \det(U^{-1})\mathrm{Vol}(B_n) = (\det(A))^{-\frac{1}{2}}\mathrm{Vol}(B_n). \qquad \square$$

Theorem 5.2 (Löwner-John) *If $K \subseteq \mathbb{R}^n$ is compact, then there is a unique ellipsoid $LJ(K)$ of minimum volume containing K. Furthermore, if K is convex, then scaling $LJ(K)$ by a factor of $\frac{1}{n}$ results in an ellipsoid which is contained in K.*

Lemma 5.3 *Let $P \subseteq \mathbb{R}^n$ be an ellipsoid centered at a and let H be a halfspace defined by a hyperplane which contains a. Then there is an ellipsoid P' with the following properties:*

(i) $P \cap H \subseteq P'$

(ii) $\mathrm{Vol}(P') \leq e^{-\frac{1}{2(n+1)}}\,\mathrm{Vol}(P)$

Proof: First consider the special case that $P = B_n$. In this case, we may assume (by possibly performing a rotation) that $H = \{x \in \mathbb{R}^n : x_1 \geq 0\}$. We claim that

$$P' = \left\{ x \in \mathbb{R}^n : \frac{(n+1)^2}{n^2}\left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\sum_{i=2}^{n} x_i^2 \leq 1 \right\}$$

has the desired properties. First, note that P' is an ellipsoid with center $(\frac{1}{n+1}, 0, \ldots, 0)$ and matrix A a diagonal matrix with $\frac{(n+1)^2}{n^2}$ in the first position and $\frac{n^2-1}{n^2}$ in the others. To prove (ii) we need to compute $\mathrm{Vol}(P')$. Let U be the diagonal matrix with $U^2 = A$. Then we have

$$\begin{aligned}
\frac{\mathrm{Vol}(P')}{\mathrm{Vol}(B_n)} &= (\det(A))^{-\frac{1}{2}} \\
&= \left(\frac{n}{n+1}\right)\left(\frac{n}{n^2-1}\right)^{\frac{n-1}{2}} \\
&\leq e^{-\frac{1}{n+1}} e^{\frac{n-1}{(n^2-1)2}} = e^{-\frac{1}{n+1}} e^{\frac{1}{2(n+1)}} = e^{-\frac{1}{2(n+1)}}
\end{aligned}$$

Here we have used the fact that $1 + x \leq e^x$ for al x. Now, to prove (i), let $x \in H \cap P$ and observe that

$$\begin{aligned}
&\frac{(n+1)^2}{n^2}\left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\sum_{i=2}^{n} x_i^2 \\
=&\frac{2(n+1)}{n^2}x_1^2 - \frac{2(n+1)}{n^2}x_1 + \frac{1}{n^2} + \frac{n^2-1}{n^2}\sum_{i=1}^{n} x_i^2 \\
\leq&\frac{2(n+1)}{n^2}x_1(x_1-1) + \frac{1}{n^2} + \frac{n^2-1}{n^2} \\
\leq&1.
\end{aligned}$$

This completes the proof in the special case when $P = B_n$. For the general case, choose an affine linear transformation $f(x) = Bx + b$ so that f maps P to B_n. Then apply the special case to choose P' and then apply the transformation f^{-1}. Since affine transformations preserve volume ratios, this yields a suitable ellipsoid for P. $\square$