1 Semidefinite Matrices

All matrices here are assumed to be real. Elements of $\mathbb{R}^n$ are column vectors, and we assume by default that square matrices are $n \times n$. We require the following two properties of a symmetric matrix A which we shall not prove.

- All eigenvalues of A are real.
- There is an orthonormal basis consisting of eigenvectors of A.

A matrix is *orthogonal* if its columns form an orthonormal basis. It follows from the second condition above that there is an orthogonal matrix U and a diagonal matrix D so that $AU = UD$. Since $U^\top U = 1$, this may be rewritten as $A = UDU^\top$. This last equation is the basic decomposition of symmetric matrices we will use.

Semidefinite & Definite: Let A be a symmetric matrix. We say that A is *(positive) semidefinite*, and write $A \succeq 0$, if all eigenvalues of A are nonnegative. We say that A is *(positive) definite*, and write $A \succ 0$, if all eigenvalues of A are positive.

Principal Minor: For a symmetric matrix A, a *principal minor* is the determinant of a submatrix of A which is formed by removing some rows and the corresponding columns.

Proposition 1.1 *For a symmetric matrix A, the following conditions are equivalent.*

(1) $A \succeq 0$.

(2) $A = U^\top U$ *for some matrix* U.

(3) $x^\top A x \geq 0$ *for every* $x \in \mathbb{R}^n$.

(4) *All principal minors of A are nonnegative.*

Proof: We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1), and then (2) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Write $A = UDU^\top$ where U is orthogonal and D diagonal. Now, the entries on the diagonal of D are the eigenvalues of A (which are nonnegative), so we may write $D = C^2$ where C is a diagonal matrix. Then we have $A = UCCU^\top = UC(UC)^\top$ as desired.

(2) $\Rightarrow$ (3): For every $x \in \mathbb{R}^n$ we have $x^\top A x = x^\top U^\top U x = (Ux)^\top (Ux) \geq 0$.

(3) $\Rightarrow$ (1): If v is an eigenvector of A with eigenvalue λ then $0 \leq v^\top Av = \lambda v^\top v$ which implies that $\lambda \geq 0$.

(2) $\Rightarrow$ (4): Let B be a submatrix of A formed by deleting the rows and columns with index in the set S. Then modify U to form V by deleting the columns in S. Now, $\det(B) = \det(V^\top V) = (\det(V))^2 \geq 0$.

(4) $\Rightarrow$ (1): We prove the contrapositive by induction. We may assume that A has a unit eigenvector v with eigenvalue $\lambda < 0$. If A has only one eigenvalue ≤ 0, then $\det(A) < 0$ and we are finished. Otherwise, choose a unit eigenvector u orthogonal to v with eigenvalue $\mu \leq 0$. Now, choose $s \in \mathbb{R}$ so that the vector $w = v + su$ has at least one zero coordinate, say the i^{th}. If A' is the matrix obtained from A by removing the i^{th} column and row and w' is obtained by removing the i^{th} coordinate of w, then we have $(w')^\top A'w' = w^\top Aw = \lambda + s^2\mu < 0$. So, A' is not semidefinite (since we have already demonstrated (1) $\Leftrightarrow$ (3)), and the result follows by applying induction to it. $\square$

Ellipsoid: If A is definite, then $\{x \in \mathbb{R}^n : x^\top Ax \leq 1\}$ is an *ellipsoid.*

Proposition 1.2 *If $A, B \succeq 0$ and $s, t \in \mathbb{R}$ satisfy $s, t \geq 0$ then $sA + tB \succeq 0$.*

Proof: Using (3) of the previous proposition, we have that for every $x \in \mathbb{R}^n$

$$x^\top (sA + tB)x = s(x^\top Ax) + t(x^\top Bx) \geq 0 \qquad \square$$

Corollary 1.3 *Another property equivalent to $A \succeq 0$ is*

(5) *There exist $x_1 \ldots x_k \in \mathbb{R}^n$ so that $A = \sum_{i=1}^k x_i x_i^\top$.*

Proof: It follows from (2) of Proposition 1.1 that $x_i x_i^\top$ is always positive semidefinite, and then from the previous proposition that any matrix satisfying (5) is semidefinite. For the other direction, suppose A is semidefinite, choose U so that $A = U^\top U$ and let x_i be the i^{th} row of U. Then $A = \sum_{i=1}^k x_i x_i^\top$ as desired. $\square$

Order: We extend $\succeq$ to a relation by defining $A \succeq B$ if $A - B \succeq 0$. Then if $A \succeq B$ and $B \succeq C$ we have $A - C = (A - B) + (B - A) \succeq 0$ so $A \succeq C$. Also $A \succeq B$ and $B \succeq A$ imply that $A = B$, so $\succeq$ defines a partial order.

Dot product: If A, B are $n \times n$ matrices we define

$$A \cdot B = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ij} = \mathrm{tr}(A^\top B).$$

(recall that $\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$).

Proposition 1.4

(i) *If $A, B \succeq 0$ then $A \cdot B \geq 0$, and further $A \cdot B = 0$ implies $AB = 0$.*

(ii) *A symmetric matrix A is semidefinite if $A \cdot B \geq 0$ for every $B \succeq 0$.*

Proof: (i): Using (2) of Proposition 1.1 to write $A = U^\top U$ and $B = V^\top V$ we have

$$\mathrm{tr}(A^\top B) = \mathrm{tr}(U^\top U V^\top V) = \mathrm{tr}(U V^\top V U^\top) = \mathrm{tr}(U V^\top (U V^\top)^\top) \geq 0.$$

If we have equality in the above expression, then $UV^\top = 0$ so $AB = U^\top U V^\top V = 0$.

(ii): Let v be an eigenvector of A with eigenvalue λ. Then we have

$$0 \leq A \cdot (vv^\top) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} v_i v_j = v^\top A v = \lambda v^\top v$$

and it follows that $\lambda \geq 0$. Thus $A \succeq 0$ as required. $\square$

Cones and Polars: A set $C \subseteq \mathbb{R}^m$ is a *cone* if for all $x, y \in C$ and all $s, t \in \mathbb{R}$ with $s, t \geq 0$ we have $sx + ty \in C$. The *polar* of C is

$$C^\circ = \{x \in \mathbb{R}^m : \ x^\top y \geq 0 \text{ for every } y \in C\}$$

Note that C° is a cone. In general every closed cone C satisfies $(C^\circ)^\circ = C$.

Proposition 1.5 *Consider the vector space consisting of all symmetric $n \times n$ matrices and let $\mathcal{C}$ be the set of all semidefinite matrices. Then*

(i) *$\mathcal{C}$ is a cone.*

(ii) *$\mathcal{C}^\circ = \mathcal{C}$.*

(iii) *The interior of $\mathcal{C}$ is the set of definite matrices.*

Proof: Proposition 1.2 is equivalent to (i), while the previous proposition is equivalent to (ii). The proof of (iii) follows from the fact that the spectrum is a continuous function of a matrix. $\square$