8 Shannon Capacity

Strong Product: If G, H are graphs, the *strong product* of G and H, denoted $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ where $(u, v) \sim (u', v')$ in $G \boxtimes H$ if u, u' are either equal or adjacent in G and v, v' are either equal or adjacent in H. We define $G^k = \underbrace{G \boxtimes G \ldots \boxtimes G}_{k}$.

Motivation: Imagine a communications channel where we have the ability to transmit n different types of signals, but some pairs of these signals are similar, and may be confused due to noise. We may model this with a graph G with vertex set the n possible signals, and two vertices joined by an edge if these signals may be confused. If we wish to arrange a protocol where no two messages can be confused, and we are to transmit one signal, then we must choose an independent set of G of possible signals to use. On the other hand, if we have k rounds of communication, then we can use an independent set of G^k of possible messages with no danger of confusion.

Shannon Capacity: The *Shannon Capacity* of G is the best effective rate of communication per round if we are free to choose k. More precisely, it is

$$\Sigma(G) = \sup_k \sqrt[k]{\alpha(G^k)} = \lim_{k \to \infty} \sqrt[k]{\alpha(G^k)}$$

Notes: The second equality above follows from $\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$ and Fekete's Lemma. Most authors use Θ for Shannon Capacity and ϑ for the Lovasz Theta Function, but we have chosen to use Σ instead to avoid having two "theta"s.

Example: Let C_5 have vertex set $\{1, 2, 3, 4, 5\}$ with two vertices adjacent if they are consecutive in the cyclic order. Then $\{(1, 1), (2, 3), (3, 5), (4, 2), (5, 4)\}$ is an independent set in $C_5 \boxtimes C_5$, so $\Sigma(C_5) \geq \sqrt{\alpha(C_5^2)} \geq \sqrt{5}$

Lemma 8.1 *If f is a real valued function on graphs which satisfies:*

(i) $\alpha(G) \leq f(G)$

(ii) $f(G \boxtimes H) \leq f(G)f(H)$

then $\Sigma(G) \leq f(G)$.

Proof: We have $\alpha(G^k) \le f(G^k) \le (f(G))^k$ and then taking k^{th} roots gives $\sqrt[k]{\alpha(G^k)} \le f(G)$ for every k, so $\Sigma(G) \le f(G)$. $\square$

Clique Cover: We define $\bar{\chi}(G)$ to be the minimum number of cliques required to cover the vertices of G, or equivalently, $\bar{\chi}(G) = \chi(\bar{G})$.

Observation 8.2 $\Sigma(G) \le \bar{\chi}(G)$ *for every graph* G.

Proof: It is immediate that $\alpha(G) \le \bar{\chi}(G)$. Since the strong product of two cliques is another clique, it follows that $\bar{\chi}(G \boxtimes H) \le \bar{\chi}(G)\bar{\chi}(H)$. Now Lemma 8.1 completes the proof. $\square$

Fractional Parameters: Let G be a graph and let A be the vertex-clique incidence matrix of G (so the rows of A are indexed by $V(G)$, the columns by the cliques of G, and there is a 1 in position (v, X) if v is in the clique X and a 0 otherwise). Then (check!)

$$\bar{\chi}(G) = \min\{1^\top x : Ax \ge 1,\ x \ge 0, \text{ and } x \text{ is integral}\}$$
$$\alpha(G) = \max\{y^\top 1 : y^\top A \le 1,\ y \ge 0, \text{ and } y \text{ is integral}\}$$

Relaxing the integrality constraints yields the following fractional parameters:

$$\bar{\chi}_f(G) = \min\{1^\top x : Ax \ge 1,\ x \ge 0\}$$
$$\alpha_f(G) = \max\{y^\top 1 : y^\top A \le 1,\ y \ge 0\}$$

Now, by LP-duality we have the following chain of inequalities:

$$\alpha(G) \le \alpha_f(G) = \bar{\chi}_f(G) \le \bar{\chi}(G)$$

Observation 8.3 $\Sigma(G) \le \bar{\chi}_f(G)$ *for every graph* G.

Proof: By the above we have that $\alpha(G) \le \bar{\chi}_f(G)$. It follows from the fact that the strong product of two cliques is a clique that $\bar{\chi}_f(G \boxtimes H) \le \bar{\chi}_f(G)\bar{\chi}_f(H)$. Now, the result follows from lemma 8.1. $\square$

Vector Representations: A *vector representation* of a graph $G = (V, E)$ is an assignment of a unit vector u_i to each vertex $i \in V$ so that $u_i \cdot u_j = 0$ whenever i, j are distinct and nonadjacent. In addition there is a unit vector c called a *handle* and the *value* of this representation is defined to be:

$$\max_{i \in V} \frac{1}{(c \cdot u_i)^2}$$

We define $\theta(G)$ to be the minimum value over all vector representations of G.

Lemma 8.4 *$\alpha(G) \leq \theta(G)$ for every graph G.*

Proof: Let $\{u_i\}_{i \in V}$ be a vector representation of $G = (V, E)$ with handle c and value $\theta(G)$. Let $X \subseteq V$ be an independent set of size $\alpha(G)$. Then (using the fact that $c = \sum_{i \in I}(b_i \cdot c) b_i$ for an orthonormal basis $\{b_i\}_{i \in I}$ and thus $c \cdot c = \sum_{i \in I}(b_i \cdot c)^2$) we have

$$\begin{aligned} 1 &= c \cdot c \\ &\geq \sum_{i \in X} (c \cdot u_i)^2 \\ &\geq \tfrac{\alpha(G)}{\theta(G)} \end{aligned}$$

which gives the desired inequality. □

Tensor Product: If $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ the *tensor product* of a an b is

$$a \otimes b = (a_1 b_1, a_1 b_2, \ldots a_1 b_m, a_2 b_1 \ldots, a_n b_m) \in \mathbb{R}^{nm}$$

Note that if $a, c \in \mathbb{R}^n$ and $b, d \in \mathbb{R}^m$ then

$$(a \otimes b) \cdot (c \otimes d) = (a \cdot c)(b \cdot d) \tag{1}$$

Lemma 8.5 *$\theta(G \boxtimes H) \leq \theta(G)\theta(H)$ for all graphs G, H.*

Proof: Let $\{u_i\}_{i \in V(G)}$ be a vector representation of G with handle c and value $\theta(G)$ and let $\{v_j\}_{j \in V(H)}$ be a vector representation of H with handle d and value $\theta(H)$. Then consider $\{u_i \otimes v_j\}_{(i,j) \in V(G \boxtimes H)}$. It follows from (1) that each of these vectors has unit length. Further, if two vertices $(i, j), (i', j') \in V(G \boxtimes H)$ are nonadjacent then either i, i' are nonadjacent in G or j, j' are nonadjacent in H so it follows from (1) that $(u_i \otimes v_j) \cdot (u_{i'} \otimes v_{j'}) = 0$. Thus, we have a vector representation of $G \boxtimes H$. Taking $c \otimes d$ as the handle we find that this representation has value at most

$$\max_{(i,j) \in V(G \boxtimes H)} \frac{1}{((c \otimes d) \cdot (u_i \otimes v_j))^2} = \max_{(i,j) \in V(G \boxtimes H)} \frac{1}{(c \cdot u_i)^2} \frac{1}{(d \cdot v_j)^2} = \theta(G)\theta(H)$$

Theorem 8.6 $\Sigma(G) \leq \theta(G)$

Proof: This follows immediately from Lemma 8.1 and the previous two lemmas. □

Theorem 8.7 $\Sigma(C_5) = \sqrt{5}$.

Proof: We showed earlier that $\Sigma(C_5) \geq \sqrt{5}$. For the upper bound, we shall construct a vector representation of C_5. To do this, consider a 5 prong umbrella which is slowly raised until nonadjacent spokes are orthogonal. Then taking unit vectors in these directions and handle the handle of the umbrella yields a vector representation of C_5 with value $\sqrt{5}$. Thus $\Sigma(C_5) \leq \theta(C_5) \leq \sqrt{5}$. $\square$